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Nonlinear Stability of Riemann Ellipsoids with Symmetric Configurations

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Abstract Using modern differential geometric methods, we study the relative equilibria for Dirichlet's model of a self-gravitating fluid mass having at least two equal axes. We show that the only relative equilibria of this type correspond to Riemann ellipsoids for which the angular velocity and vorticity are parallel to the same principal axis of the body configuration. The two solutions found are MacLaurin and transversal spheroids.

The singular reduced energy-momentum method developed in Rodríguez-Olmos (Nonlinearity 19(4):853–877, 2006) is applied to study their nonlinear stability and instability. We found that the transversal spheroids are nonlinearly stable for all eccentricities while for the MacLaurin spheroids, we recover the classical results. Comparisons with other existing results and methods in the literature are also made.

Keywords Riemann ellipsoids · Hamiltonian dynamics · Momentum maps · Nonlinear stability · Relative equilibria

Mathematics Subject Classification (2000) 70H14 · 70H33 · 37J25

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1 Introduction

A Riemann ellipsoid is a relative equilibrium for a dynamical model of a rotating self-gravitating fluid mass that remains an ellipsoid at all times, and for which the fluid velocity is a linear function of the coordinates. This model, first studied and formulated by Dirichlet, is nowadays known as Dirichlet's model. The linearity assumption on the allowed motions makes the study of these deformable bodies very attractive since it implies that their dynamics is governed by a system of ordinary differential equations with a finite number of degrees of freedom. These bodies are also known as affine-rigid bodies or pseudo-rigid bodies. Dirichlet's model can be viewed as a first order model for the study of the shape of the Earth, and the study of the stability of its solutions has been used in planetary stability research (see, for instance Todhunter 1873).

The study of self-gravitating fluid masses has a long history which can be traced back to Newton's times with many contributions from Dirichlet, MacLaurin, Jacobi, Dedekind, Riemann, Liapunov, Poincaré, and Cartan, just to name a few. In 1861, Riemann published a remarkable paper (Riemann 1861) where he reformulates the equations of the motion given by Dirichlet, determines conditions (known nowadays as Riemann's theorem) for the existence of relative equilibria and studies their stability using an energy criterion.

We can distinguish two classical approaches to the study of the stability of Riemann ellipsoids. One initiated independently in the latter part of the nineteenth century by Poincaré (1885) and Liapunov (1904) who used expansions in spherical and ellipsoidal harmonics to study the stability of MacLaurin ellipsoids and Jacobi ellipsoids, not only under Dirichlet's assumptions, but also under more general conditions (allowing perturbations not preserving the ellipsoidal shape). The other approach occurred in the middle of the twentieth century with the works of Chandrasekhar and collaborators who developed the so-called virial method by applying it to the study of the linear stability of Riemann ellipsoids. These works are collected in the book of Chandrasekhar (1987) which constitutes a comprehensive survey on the subject and related problems with many historical facts on this model.

In recent times, the subject has had the attention of several researchers, in particular in what respects to the application of new formulations and methods (Rosensteel 1998, 2001) to study rotating deformable bodies, not only subjected to the self-gravitating potential, but also for other potentials modeling nuclei (see, for instance Rosensteel 1988), or elastic bodies (see Cohen and Muncaster 1988 and Lewis and Simo 1990).

Our aim is to use geometric methods not only to obtain a complete characterization of the conditions for the existence of Riemann ellipsoids having configurations with at least two equal axes (symmetric configurations), but also to obtain the complete description of their nonlinear stability. These geometric methods exploit the geometry and symmetries of the problem and its Hamiltonian structure. Some works using the same philosophical approach to Dirichlet's model are available in the literature for studying several aspects of the problem such as in Roberts and Sousa-Dias (1999) which obtains Riemann's theorem as a consequence of the symmetry alone, or Lewis (1993) where the stability (and not the instability) of the MacLaurin spheroids is approached using the Lagrangian block diagonalization method (see Rodríguez-Olmos

2006 for the comparison of this method with the singular reduced energy-momentum method).

We do not address the problem of the stability for Riemann ellipsoids with three distinct axes for which the self-gravitating potential is an elliptic integral. In Fassò and Lewis (2001), numerical analysis techniques have been employed to study stability for finite but long time scales of Riemann ellipsoids with three distinct axes. These authors do not address the stability of Riemann ellipsoids with symmetric configurations considering an open problem the nonlinear stability of this type of spheroids other than those of MacLaurin (see the conclusions section of Fassò and Lewis 2001).

We view Dirichlet's model as a Hamiltonian system where Hamilton's function, h, has the form kinetic plus potential energy and is defined on the cotangent bundle, $T^*\mathrm{SL}(3)$, of the set of all 3×3 matrices of determinant 1. Furthermore, h is invariant for the action of $G = \mathbb{Z}_2 \ltimes (\mathrm{SO}(3) \times \mathrm{SO}(3))$ on the phase space. The $\mathrm{SO}(3) \times \mathrm{SO}(3)$ symmetry reflects the existence of two conserved vector quantities (by Noether's theorem): the angular momentum and circulation. The $\mathbb{Z}_2$ symmetry reflects the reciprocity theorem of Dedekind: Interchanging the angular velocity and vorticity vectors, one obtains another (physically different) solution for Dirichlet's model with the same geometric configuration.

It is well known that Noether's conserved quantities are organized as the components of the momentum map. In the case of relative equilibria with configurations having at least two equal axes (symmetric configurations), the corresponding momentum map value can be singular. Recently, one of the authors developed in Rodríguez-Olmos (2006) a method appropriate for the study of nonlinear stability of this kind of relative equilibria. This construction extends the so-called reduced energy-momentum method of Simo et al. (1991) to the case of singular relative equilibria and so we will refer to the method (Rodríguez-Olmos 2006) as the singular reduced energy-momentum method. This is the approach we use in this work to study the nonlinear stability and instability of Riemann ellipsoids with symmetric configurations.

The preliminary sections of the paper are organized as follows: In Sect. 2, we give the geometric formulation of Dirichlet's model. In Sect. 3, the singular reduced energy-momentum method is briefly reviewed, and in Sect. 4, we compute the augmented potential energy for symmetric configurations, as a necessary step toward the stability analysis.

The main results of the paper are in the following two sections. In Sect. 5, Theorem 5.1, we give the complete characterization of all the possible Riemann ellipsoids with symmetric configurations. We prove that for Dirichlet's model the only relative equilibria with configurations having at least two equal axes are: the spherical configuration which has zero angular velocity and vorticity, the *MacLaurin spheroids* which are oblate spheroids rotating around the (shortest) symmetry axis and have angular velocity and vorticity aligned with it, and the *transversal spheroids*, which have prolate spheroidal configurations that rotate around an axis, say $\mathbf{n}$, perpendicular to the (longest) symmetry axis and have angular velocity and vorticity aligned with $\mathbf{n}$. We also prove that there are no symmetric relative equilibria for which the angular velocity and vorticity are not aligned with a principal axis of the relative equilibrium configuration. That is, there are no symmetric configurations which are not of type S in Chandrasekhar's terminology.

In Sect. 6, we apply the singular reduced energy-momentum method to the study of the nonlinear stability of the relative equilibria found in the previous section. The main results of this section are Theorems 6.1, 6.2, and 6.3 giving respectively necessary and sufficient conditions for the nonlinear stability of the spherical equilibrium, MacLaurin spheroids, and transversal spheroids.

Our results on the eccentricity range for the nonlinear G_μ-stability of the MacLaurin spheroids agree with those already obtained by Riemann (see Remark 6.1) and confirmed by Liapunov and Poincaré who studied the problem of self-gravitating fluid masses under hypotheses different from Dirichlet's ones (see Remark 6.1(3)). In the works of these authors, we have not found reference to the transversal spheroids, however, their existence is acknowledged in page 143 of Chandrasekhar's book (Chandrasekhar 1987). See also Remarks 5.3 and 6.3.

In conclusion, this work presents, from a purely geometric point of view, a self-contained and complete study of the nonlinear stability of all symmetric relative equilibria for Dirichlet's model. At the same time, the richness of the model helps to clarify the applicability of the singular reduced energy-momentum method, providing also a methodology for its use in other models.

2 Geometric Formulation of Dirichlet's Model

Let $(M, \langle\langle\cdot,\cdot\rangle\rangle)$ be a Riemannian manifold (the configuration manifold), G a Lie group that acts by isometries on M (the symmetry group) and $V \in C^\infty(M)$ a G-invariant function (the potential energy). With these ingredients, we construct a symmetric Hamiltonian system on T^*M (which is a manifold equipped with a natural symplectic structure) as follows: The potential energy V can be lifted to T^*M with the pullback of the cotangent bundle projection $\tau : T^*M \to M$. We denote this lifted function also by V. The Riemannian metric on M induces an inner product on each cotangent fiber T_x^*M, $x \in M$. Then the Hamiltonian is defined as

$$h(p_x) = \frac{1}{2}\|p_x\|^2 + V(x), \qquad p_x \in T_x^*M.$$

The G-action on M induces a cotangent-lifted Hamiltonian action on T^*M with associated equivariant momentum map $\mathbf{J} : T^*M \to \mathfrak{g}^*$ defined by

$$\langle \mathbf{J}(p_x), \xi \rangle = \langle p_x, \xi_M(x) \rangle \quad \forall \xi \in \mathfrak{g},$$

where ξ_M is the fundamental vector field on M associated to the generator ξ, defined by

$$\xi_M(x) = \left.\frac{d}{dt}\right|_{t=0} e^{t\xi} \cdot x.$$

This momentum map is Ad^*-equivariant in the sense that $\mathbf{J}(g \cdot p_x) = \mathrm{Ad}^*_{g^{-1}} \mathbf{J}(p_x)$ for every $p_x \in T_x^*M$, $g \in G$.

The Hamiltonian h is G-invariant for this lifted action (this follows from the invariance of the metric and of V). Therefore, due to Noether's theorem, the components of $\mathbf{J}$ are conserved quantities for the Hamiltonian dynamics associated to h. The quadruple $(M, \langle\langle\cdot,\cdot\rangle\rangle, G, V)$ is called a *symmetric simple mechanical system*.

Let $(M, \langle\langle\cdot,\cdot\rangle\rangle, G, V)$ be a simple mechanical system with symmetry. A *relative equilibrium* is a point in phase space $p_x \in T^*M$ such that its Hamiltonian orbit lies inside a group orbit for the cotangent-lifted action. This amounts to the existence of a generator $\xi \in \mathfrak{g}$ such that the evolution of p_x is given by $e^{t\xi} \cdot p_x$. The element ξ is called a *velocity* for the relative equilibrium and is defined up to addition of elements in $\mathfrak{g}_{p_x} = \mathrm{Lie}(G_{p_x})$, where G_{p_x} is the stabilizer of p_x under the cotangent-lifted action. A useful criterion for finding relative equilibria in simple mechanical systems is given by the following theorem:

Theorem 2.1 (Marsden 1992) *A point $p_x \in T^*M$ of a symmetric simple mechanical system $(M, \langle\langle\cdot,\cdot\rangle\rangle, G, V)$ is a relative equilibrium with velocity $\xi \in \mathfrak{g}$ if and only if the following conditions are verified*:

1. $p_x = \langle\langle \xi_M(x), \cdot\rangle\rangle$.
2. *x is a critical point of* $V_\xi := V - \frac{1}{2}\langle \xi, \mathbb{I}(\cdot)(\xi)\rangle$,

where $\mathbb{I} : M \times \mathfrak{g} \to \mathfrak{g}^*$ *is defined by* $\langle \xi, \mathbb{I}(x)(\eta)\rangle = \langle\langle \xi_M(x), \eta_M(x)\rangle\rangle$. *Moreover, the momentum* $\mu = \mathbf{J}(p_x) \in \mathfrak{g}^*$ *of the relative equilibrium is given by* $\mu = \mathbb{I}(x)(\xi)$.

Note that in virtue of the above theorem, any relative equilibrium is characterized by a configuration-velocity pair $(x, \xi) \in M \times \mathfrak{g}$ satisfying $\mathbf{d}V_\xi(x) = 0$. The map $\mathbb{I}$ is called the *locked inertia tensor*, while the function V_ξ is called the *augmented potential*. We indicate for later use that the kernel of $\mathbb{I}$ is precisely $\mathfrak{g}_x$, the Lie algebra of G_x, the stabilizer of x for the G-action on M. The knowledge of the pair (x, ξ) allows us to compute the stabilizer of the corresponding relative equilibrium $p_x = \langle\langle \xi_M(x), \cdot\rangle\rangle$ with the formula (see Rodríguez-Olmos and Sousa-Dias 2002):

$$G_{p_x} = \{g \in G_x \,:\, \mathrm{Ad}_g\xi - \xi \in \mathfrak{g}_x\}.$$

Dirichlet's model is a symmetric simple mechanical system for the motion of a homogenous and incompressible fluid mass of density ρ having as reference configuration the unit ball centered at the origin in $\mathbb{R}^3$ and subjected to the self-gravitating potential. The only allowed configurations for this model are linear embeddings of the reference ball into $\mathbb{R}^3$ preserving volume and orientation. The configuration manifold M for a self-gravitating fluid mass under Dirichlet's conditions is then SL(3), the group of all 3×3 matrices with determinant equal to 1, which is equivalent to the space of orientation and volume preserving linear automorphisms of $\mathbb{R}^3$. In what follows, we review the geometric formulation of Dirichlet's model as a symmetric simple mechanical system on $GL^+(3)$, the group of all 3×3 matrices with positive determinant, with a symmetric holonomic constraint.

The singular value decomposition of any linear automorphism of $\mathbb{R}^3$ allows to decompose (nonuniquely) any matrix $F \in \mathrm{GL}^+(3)$ as

$$F = LAR^T,$$

where $L, R \in \mathrm{SO}(3)$, and A is a diagonal matrix with positive entries called singular values (the square roots of the eigenvalues of $C = F^T F$). It follows from this decomposition that the reference unit ball is mapped by F into a solid ellipsoid of equation $\mathbf{X} \cdot C^{-1}\mathbf{X} = 1, \mathbf{X} \in \mathbb{R}^3$, having principal axes half-lengths equal to the entries of A. Physically, the matrix L describes the rigid rotation of the body in space relative to an inertial frame and R is related to the rigid internal motion of the fluid with respect to a moving frame. Then A is an orientation preserving dilatation of the original reference body into an ellipsoid with principal axes aligned with the eigenvectors of A. The condition on the volume preservation of the total embedding corresponds to impose the holonomic constraint $\det F = 1$ (or equivalently $\det A = 1$), which in turn amounts to consider our system as defined on SL(3).

The tangent space at $F \in \mathrm{GL}^+(3)$ is isomorphic to L(3), the vector space of 3×3 matrices. We can define a Riemannian metric on $\mathrm{GL}^+(3)$ as

$$\langle\langle \delta F_1, \delta F_2 \rangle\rangle = T \,\mathrm{tr}\big(\delta F_1^T \delta F_2\big), \tag{1}$$

for $\delta F_1, \delta F_2 \in T_F \mathrm{GL}^+(3)$, and T is a constant depending on the density of the reference body and other physical parameters of the system. In the case of interest here, the reference body is a homogeneous unit ball of constant density ρ, and T in (1) is

$$T = \frac{4\pi}{15}\rho.$$

The symmetry group G of our model is the semidirect product $G = \mathbb{Z}_2 \ltimes (\mathrm{SO}(3) \times \mathrm{SO}(3))$, where $\mathbb{Z}_2 = \{e, \sigma\}$. Several actions of G of interest in this paper are:

(1) The G-action on G: If $(\gamma; g, h), (\gamma'; g', h') \in \mathbb{Z}_2 \ltimes (\mathrm{SO}(3) \times \mathrm{SO}(3))$ then

$$(\gamma; g, h) \cdot \big(\gamma'; g', h'\big) = \big(\gamma\gamma'; (g, h) \cdot \big(\gamma \cdot \big(g', h'\big)\big)\big),$$

where for the nontrivial element $\sigma \in \mathbb{Z}_2$, $\sigma \cdot (g', h') = (h', g')$, and $\mathrm{SO}(3) \times \mathrm{SO}(3)$ acts on itself by the direct product of left matrix multiplications.

(2) The adjoint representation of G: We identify the Lie algebra of SO(3), the set $\mathfrak{so}(3)$ of skew-symmetric 3×3 matrices, with $\mathbb{R}^3$ via the usual isomorphism $\widehat{\ }: \mathbb{R}^3 \to \mathfrak{so}(3)$:

$$v = (v_1, v_2, v_3) \mapsto \widehat{v} = \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix}. \tag{2}$$

This is an isomorphism of Lie algebras, i.e. $(\mathfrak{so}(3), [\cdot, \cdot])$ is isomorphic by (2) to $(\mathbb{R}^3, \times)$, where $[\ ,\ ]$ denotes the commutator of matrices and $\times$ denotes the vector product of vectors in $\mathbb{R}^3$.

The Lie algebra of G is then $\mathfrak{g} = \mathbb{R}^3 \oplus \mathbb{R}^3$ and the adjoint action is given by

$$\mathrm{Ad}_{(\gamma; g, h)}(\xi_L, \xi_R) = \gamma \cdot (g \cdot \xi_L, h \cdot \xi_R),$$

where $\sigma \cdot (\xi_L, \xi_R) = (\xi_R, \xi_L)$ and $g \cdot \xi_L$ is the rotation of ξ_L by g (and similarly for ξ_R).

Using the standard inner product in $\mathbb{R}^3$, we also identify $\mathfrak{g}^*$ with $\mathbb{R}^3 \oplus \mathbb{R}^3$. Under this identification, it follows easily that the coadjoint representation has the expression

$$\mathrm{Ad}^*_{(\gamma;g,h)^{-1}}(\mu_L, \mu_R) = \gamma \cdot (g \cdot \mu_L, h \cdot \mu_R).$$

(3) The G-action on $\mathrm{GL}^+(3)$:

$$(e; L, R) \cdot F = LFR^T, \qquad (\sigma; L, R) \cdot F = RF^T L^T.$$

Note that the $\mathbb{Z}_2$ transposition symmetry on SL(3) (first noticed by Dedekind) maps a rigidly rotating configuration without internal motion into one that is stationary in space, but with rigid fluid internal motions. That is, for a given ellipsoid there is an adjoint one, obtained by transposition. These adjoint type of ellipsoids are called Dedekind ellipsoids. More generally, the transposition symmetry interchanges external rotations and internal motions for any solution of Dirichlet's model.

Any G-invariant function f on $\mathrm{GL}^+(3)$ can be written as

$$f(F) = \tilde{f}\big(I_1(F), I_2(F), I_3(F)\big),$$

where $\tilde{f} : \mathbb{R}^3 \to \mathbb{R}$, and I_1, I_2, and I_3 are the three principal invariants of a 3×3 matrix, given by

$$I_1(F) = \mathrm{tr}(S), \tag{3}$$

$$I_2(F) = \frac{1}{2}\big(\mathrm{tr}^2(S) - \mathrm{tr}\big(S^2\big)\big), \tag{4}$$

$$I_3(F) = \det(S), \tag{5}$$

with $S = FF^T$ (also valid interchanging S with $C = F^T F$). Note that I_1, I_2, I_3 are G-invariant and that 1 is a regular value of I_3. Hence, we have $\mathrm{SL}(3) = I_3^{-1}(1)$ as a G-invariant submanifold of $\mathrm{GL}^+(3)$. The condition $I_3 = 1$ is the holonomic constraint of the model. Note also that the restriction of a G-invariant function $f \in C^G(\mathrm{GL}^+(3))$ to SL(3) is given by

$$f(F) = \tilde{f}\big(I_1(F), I_2(F), 1\big),$$

for $F \in \mathrm{SL}(3) \subset \mathrm{GL}^+(3)$. Therefore, any G-invariant function on SL(3) can be written as $h(F) = \widehat{h}(I_1(F), I_2(F))$, with $\widehat{h} : \mathbb{R}^2 \to \mathbb{R}$. Any G-invariant function on SL(3) can be then extended invariantly to $\mathrm{GL}^+(3)$ by declaring it to be independent of $I_3(F)$. From now on, we will drop the tildes and hats from the corresponding functions unless there is risk of confusion.

Since G acts on $\mathrm{GL}^+(3)$ by isometries with respect to (1), the induced metric on SL(3) (which we will denote by the same symbol) is also G-invariant. For later use, we recall that tangent vectors to SL(3) can be seen as tangent vectors to $\mathrm{GL}^+(3)$ satisfying the linearization of the constraint $I_3 = 1$. In other words,

$$T_F\mathrm{SL}(3) = \big\{\delta F \in \mathrm{L}(3) : \mathrm{tr}\big(F^{-1}\delta F\big) = 0\big\}.$$

Dirichlet's model (see Chaps. 1 and 4 of (Chandrasekhar 1987) for a survey of the subject) is governed by a Hamiltonian function of the form kinetic plus potential energy on the phase space $\mathcal{P} = T^*\mathrm{SL}(3)$ given by

$$h(p_F) = \frac{T}{2}\|p_F\|^2 + V(F), \quad p_F \in T_F^*\mathrm{SL}(3).$$

Here $\|p_F\|$ is the norm of the covector p_F (seen as a 3×3 matrix) relative to the metric on SL(3). The potential energy V for a self-gravitating body of homogeneous density ρ under Dirichlet's assumptions is given by restricting the function

$$V(F) = -R\int_0^\infty \frac{ds}{\Delta(F)}, \tag{6}$$

where $F \in \mathrm{SL}(3)$, $R = \frac{8}{15}\pi^2 G\rho^2$, G is the gravitational constant and

$$\Delta(F) = \sqrt{s^3 + I_1(F)s^2 + I_2(F)s + 1}. \tag{7}$$

The quadruple $(\mathrm{SL}(3), \langle\langle\cdot,\cdot\rangle\rangle, \mathbb{Z}_2 \ltimes (\mathrm{SO}(3)\times\mathrm{SO}(3)), V)$ defines a symmetric simple mechanical system on SL(3).

The infinitesimal generator for the G-action on $M = \mathrm{GL}^+(3)$ (and on SL(3)) corresponding to $\xi = (\xi_L, \xi_R) \in \mathbb{R}^3 \times \mathbb{R}^3$ is:

$$\xi_M(F) = \frac{d}{dt}\bigg|_{t=0} \left(\exp t\widehat{\xi_L}, \exp t\widehat{\xi_R}\right)\cdot F = \widehat{\xi_L}F - F\widehat{\xi_R}. \tag{8}$$

The vectors ξ_L and ξ_R are respectively the angular velocity and vorticity of the fluid motion. We denote the momentum value of p_F by $\mathbf{J}(p_F) = (\mathfrak{j}, \mathfrak{c})$. The components $\mathfrak{j}$ and $\mathfrak{c}$ are respectively the angular momentum and circulation of the instantaneous state p_F (see, for instance Roberts and Sousa-Dias 1999 or Chandrasekhar 1987).

A Riemann ellipsoid (a.k.a. an ellipsoidal figure of equilibrium) is a solution of the Hamiltonian system defined by Dirichlet's model with angular velocity, vorticity, and principal axes lengths all constant. In our setting, Riemman ellipsoids correspond exactly to relative equilibria of the underlying symmetric simple mechanical system. Therefore, a Riemann ellipsoid is represented by a triple (F, ξ_L, ξ_R), where $F \in \mathrm{SL}(3)$ is the configuration matrix and the Lie algebra element $(\xi_L, \xi_R) \in \mathfrak{g}$ is the angular velocity-vorticity pair.

3 The Singular Reduced Energy-Momentum Method

In the last years, there are several works studying the stability of relative equilibria of Hamiltonian systems by exploiting their symmetry and the geometric properties of their phase space (see, for instance Arnold 1966, Patrick 1992, and Marsden 1992 for a overview). Most of these methods can be used to test the stability of relative equilibria lying in singular level sets of the momentum map, for instance (Lerman and Singer 1998) and (Ortega and Ratiu 1999) under the hypothesis of G_μ compact and (Montaldi 1997), Patrick et al. (2004) using topologic properties. This observation is important since, as we will see, the class of symmetric Riemann ellipsoids known as MacLaurin spheroids corresponds precisely to nontrivial relative equilibria for Dirichlet's model having singular momentum values. We refer the reader to Patrick et al. (2004) for a comparison of the applicability of several existing methods.

The generally adopted notion of stability for relative equilibria of symmetric Hamiltonian systems is that of G_μ-stability, introduced in Patrick (1992) and that we now review in the context of symmetric simple mechanical systems. This notion is closely related to the Liapunov stability of the induced Hamiltonian system on the reduced phase space.

Definition 3.1 Let $(M, \langle\langle\cdot,\cdot\rangle\rangle, G, V)$ be a symmetric simple mechanical system and $p_x \in T^*M$ a relative equilibrium with momentum value $\mu = \mathbf{J}(p_x)$. We say that p_x is G_μ stable if for every G_μ-invariant neighborhood $U \subset T^*M$ of the orbit $G_\mu \cdot p_x$ there exists a neighborhood O of p_x such that the Hamiltonian evolution of O lies in U for all time.

In the early 1990s (Simo et al. 1991), a tool known as the reduced energy-momentum method has been developed, providing sufficient conditions for the stability of relative equilibria of a simple mechanical system under the hypothesis that its momentum is a regular value of the momentum map. This method is especially well suited for simple mechanical systems since it incorporates all of their distinguishing characteristics with respect to general Hamiltonian systems. Recently, based on the characterization (Perlmutter et al. 2008) of the so-called symplectic normal space N for a cotangent-lifted action, the reduced energy-momentum method was generalized in Rodríguez-Olmos (2006) to cover also the case of singular momentum values.

In this section, we outline the implementation of this singular reduced energy-momentum method following (Rodríguez-Olmos 2006). Our setup will be as in Definition 3.1 and Sect. 2. In particular, we will fix a relative equilibrium p_x with configuration-velocity pair (x, ξ) and momentum μ. We will also assume that the G-action on M is proper and that there exists a G_μ-invariant complement to $\mathfrak{g}_\mu$ in $\mathfrak{g}$. These last two conditions are always satisfied for any relative equilibrium in Dirichlet's model due to the compactness of G.

We start by stating some key observations: First, by equivariance of $\mathbf{J}: T^*M \to \mathfrak{g}^*$ and $\tau: T^*M \to M$, one has $G_{p_x} \subset G_x$ and $G_{p_x} \subset G_\mu$. In fact, it is not difficult to prove the characterization

$$G_{p_x} = G_x \cap G_\mu. \tag{9}$$

We remark that the above formula is not valid in general for covectors p_x other than relative equilibria.

Second, also by equivariance of τ together with the bifurcation lemma (see Arms et al. 1980), if μ is a singular momentum value, then $\mathfrak{g}_x \neq \{0\}$, in which case $\mu \in (\mathfrak{g}_x)^\circ$. Third, the properness of the G-action implies that G_{p_x} is compact. This, together with (9) allows to define the following G_{p_x}-invariant splittings:

$$\mathfrak{g}_\mu = \mathfrak{g}_{p_x} \oplus \mathfrak{p} \quad \text{and} \quad \mathfrak{g} = \mathfrak{g}_x \oplus \mathfrak{p} \oplus \mathfrak{t}, \tag{10}$$

which by duality induce the splittings

$$\mathfrak{g}_\mu^* = \mathfrak{g}_{p_x}^* \oplus \mathfrak{p}^* \quad \text{and} \quad \mathfrak{g}^* = \mathfrak{g}_x^* \oplus \mathfrak{p}^* \oplus \mathfrak{t}^*. \tag{11}$$

Here $\mathfrak{p}$ and $\mathfrak{t}$ must be chosen in such a way that $\mathbb{I}(x)(\mathfrak{p}) \subset \mathfrak{t}^\circ$. Note, from the definition of the locked inertia tensor in Theorem 2.1, that $\ker \mathbb{I}(x) = \mathfrak{g}_x$, so the restriction

$$\widehat{\mathbb{I}}_0 = \mathbb{I}(x)|_{\mathfrak{p}\oplus\mathfrak{t}} : \mathfrak{p} \oplus \mathfrak{t} \to (\mathfrak{g}_x)^\circ = \mathfrak{p}^* \oplus \mathfrak{t}^*$$

is a G_{p_x}-equivariant isomorphism. Then the condition on the above splitting is that $\mathfrak{p}$ and $\mathfrak{t}$ must be orthogonal with respect to the inner product on $\mathfrak{p} \oplus \mathfrak{t}$ induced by $\widehat{\mathbb{I}}_0$.

We will denote generically the linear projections associated to the splittings (10) and (11) by the letter $\mathbb{P}$ with an appropriate subindex. For instance, $\mathbb{P}_\mathfrak{p} : \mathfrak{g} \to \mathfrak{p}$ or $\mathbb{P}_{\mathfrak{t}^*} : \mathfrak{g}^* \to \mathfrak{t}^*$. It is a consequence of Noether's theorem that $\xi \in \mathfrak{g}_\mu$ and so we will denote by $\xi^\perp = \mathbb{P}_\mathfrak{p}(\xi) \in \mathfrak{p}$ (the orthogonal velocity of the relative equilibrium).

In this work, we only need a particular version of the singular reduced energy-momentum method. Consider the following definitions:

(1) Let $\mathbf{S}$ be the linear orthogonal slice for the G-action on M at x, i.e.,

$$\mathbf{S} = \big(T_x(G\cdot x)\big)^\perp \in T_x M.$$

(2) Define the subspace $\mathfrak{q}^\mu \subset \mathfrak{g}$ as

$$\mathfrak{q}^\mu = \big\{\gamma \in \mathfrak{t} : \mathbb{P}_{\mathfrak{g}_x^*}\big[\mathrm{ad}_\gamma^* \mu\big] = 0\big\}.$$

(3) Define the space of internal variations

$$\Sigma_{\text{int}} = \big\{\gamma_M^a + a : \gamma^a \in \mathfrak{q}^\mu, a \in \mathbf{S}, \text{ and } \big(\mathbf{D}\mathbb{I}\cdot\big(\gamma_M^a(F)+a\big)\big)\big(\xi^\perp\big) \in \mathfrak{p}^*\big\}. \tag{12}$$

(4) For any $v_1, v_2 \in T_x M$, the correction term is the bilinear form on $T_x M$ defined by

$$\mathrm{corr}_\xi(x)(v_1,v_2) = \big\langle \mathbb{P}_{\mathfrak{p}^*\oplus\mathfrak{t}^*}\big[(\mathbf{D}\mathbb{I}\cdot v_1)(\xi)\big], \widehat{\mathbb{I}}_0^{-1}\big(\mathbb{P}_{\mathfrak{p}^*\oplus\mathfrak{t}^*}\big[(\mathbf{D}\mathbb{I}\cdot v_2)(\xi)\big]\big)\big\rangle. \tag{13}$$

(5) The Arnold form $\mathrm{Ar} : \mathfrak{q}^\mu \times \mathfrak{q}^\mu \to \mathbb{R}$ is defined by:

$$\mathrm{Ar}(\gamma_1,\gamma_2) = \big\langle \mathrm{ad}_{\gamma_1}^*\mu, \widehat{\mathbb{I}}_0^{-1}(\mathrm{ad}_{\gamma_2}\mu) + \mathbb{P}_{\mathfrak{p}\oplus\mathfrak{t}}\big[\mathrm{ad}_{\gamma_2}\big(\widehat{\mathbb{I}}_0^{-1}\mu\big)\big]\big\rangle. \tag{14}$$

The following theorem (Corollary 6.2 of Rodríguez-Olmos 2006) is the synthesis of the singular reduced energy-momentum method.

Theorem 3.1 *Let $p_x \in T^*M$ be a relative equilibrium with configuration-velocity pair $(x,\xi) \in M \times \mathfrak{g}$ and momentum $\mu \in \mathfrak{g}^*$ such that* $\dim(G\cdot F) < \dim M$. *Let $\xi^\perp = \mathbb{P}_\mathfrak{p}(\xi)$ be its orthogonal velocity. If the Arnold form is nondegenerate at p_x and $(\mathbf{d}_x^2 V_{\xi^\perp} + \mathrm{corr}_{\xi^\perp}(x))|_{\Sigma_{\text{int}}}$ is positive definite, then the relative equilibrium is G_μ-stable.*

When the Arnold form is nondegenerate, it is also shown in Rodríguez-Olmos (2006) that the symplectic matrix of the symplectic normal space at p_x has a particularly simple block-diagonal expression. We quote this result which will be essential in the proof of the linear instability of some Riemann ellipsoids.

Theorem 3.2 *If the Arnold form is nondegenerate at the relative equilibrium p_x with configuration-velocity pair (x,ξ) and momentum μ, the symplectic normal space at p_x, is symplectomorphic to $N = \mathfrak{q}^\mu \oplus \Sigma_{\mathrm{int}} \oplus \mathbf{S}^*$ equipped with the symplectic matrix*

$$\begin{array}{cc} & \begin{array}{ccc} \mathfrak{q}^\mu & \Sigma_{\mathrm{int}} & \mathbf{S}^* \end{array} \\ \omega_N = & \begin{bmatrix} \Xi & -\Psi & 0 \\ \Psi^T & -\mathbf{d}\chi^{\xi^\perp}\big|_{\Sigma_{\mathrm{int}}} & \mathbf{1} \\ 0 & -\mathbf{1} & 0 \end{bmatrix}, \end{array}$$

where

$$\Xi(\gamma_1,\gamma_2) = -\langle \mu, \mathrm{ad}_{\gamma_1}\gamma_2\rangle, \qquad \Psi\big(\gamma, \big(\gamma_M^b(x)+b\big)\big) = \big\langle \mu, \mathrm{ad}_\gamma \gamma^b\big\rangle$$

and $\chi^{\xi^\perp}$ is the one-form defined by $\chi^{\xi^\perp}(X) = \langle\langle \xi_M^\perp, X\rangle\rangle$, for all $X \in \mathfrak{X}(M)$.

Remark 3.1 The G-invariance of V and $\langle\langle\cdot,\cdot\rangle\rangle$ imply the following property. If (x,ξ) is a relative equilibrium, then the orbit $(g\cdot x, \mathrm{Ad}_g\xi)$ for every $g \in G$ consists of relative equilibria with the same stability or instability properties.

Remark 3.2 The reason why in the previous section we looked at Dirichlet's model as a simple mechanical system holonomically constrained is that the unconstrained space $\mathrm{GL}^+(3)$ is an open domain of the vector space $\mathrm{L}(3)$, and then the implementation of the reduced energy-momentum method is easier than if one is working directly on $\mathrm{SL}(3)$. In view of the survey of the method, the strategy will be to use the trivial extension of the self-gravitating potential to $\mathrm{GL}^+(3)$ and consider its augmented potential with respect to the locked inertia tensor corresponding to the original Riemannian metric on $\mathrm{GL}^+(3)$. Then we further augment this augmented potential with the constraint function I_3 and Lagrange multiplier λ. Denoting by

$$V^\lambda_{\xi_L,\xi_R} = V_{\xi_L,\xi_R} - \lambda \det$$

the resulting twice augmented potential, we have:

(1) Relative equilibria for Dirichlet's model correspond to triples (F,ξ_L,ξ_R) with $F \in \mathrm{GL}^+(3)$, $(\xi_L,\xi_R) \in \mathbb{R}^3 \times \mathbb{R}^3$ such that the following two equations hold:

$$\mathbf{d}V^\lambda_{\xi_L,\xi_R}(F) = 0, \quad \text{and} \quad \det(F) = 1. \tag{15}$$

(2) The stability test now follows from the following observation. If we call $\Sigma_{\mathrm{int}}^{\mathrm{GL}^+(3)}$ and $\Sigma_{\mathrm{int}}^{\mathrm{SL}(3)}$ the spaces of internal variations for $\mathrm{GL}^+(3)$ and $\mathrm{SL}(3)$ associated to

the triple (F, ξ_L, ξ_R), according to (12), we notice that

$$\Sigma_{\mathrm{int}}^{\mathrm{SL}(3)} = \Sigma_{\mathrm{int}}^{\mathrm{GL}^+(3)} \cap \ker T_F I_3.$$

Therefore, according to the general method, and standard Lagrange multiplier theory, to conclude stability it suffices to study the eigenvalues of the bilinear form

$$\left(\mathbf{d}_F^2 V^{\lambda}_{(\xi_L,\xi_R)^\perp} + \mathrm{corr}_{(\xi_L,\xi_R)^\perp}(F)\right)\Big|_{\Sigma_{\mathrm{int}}^{\mathrm{SL}(3)}},$$

where $\Sigma_{\mathrm{int}}^{\mathrm{SL}(3)}$ is seen as a vector subspace of $\Sigma_{\mathrm{int}}^{\mathrm{GL}^+(3)}$. From now on, we will omit the superindex SL(3) for the space of internal variations.

4 The Augmented Self-Gravitating Potential for Symmetric Configurations

In this section, we compute the augmented potential

$$V_{\xi_L,\xi_R} = V - \frac{1}{2}\big\langle (\xi_L, \xi_R), \mathbb{I}(F)(\xi_L, \xi_R)\big\rangle$$

in the unconstrained configuration space $\mathrm{GL}^+(3)$ and collect some results for the self-gravitating potential V. The potential $V(F)$ at a typical configuration is an elliptic integral except for symmetric configurations (i.e., with at least two equal singular values) for which it can be integrated by elementary functions. The extension to the unconstrained space of the potential V depends on $F \in \mathrm{GL}^+(3)$ through the two principal invariants I_1 and I_2 defined by (3) and (4), respectively. In the study of the existence and stability of relative equilibria of simple mechanical systems, it will be necessary to compute the first and second derivatives of V_{ξ_L,ξ_R} and the results of this section are essential to this end. In the following, we will restrict ourselves to diagonal configurations only. There is no loss of generality with this assumption since, according to the singular value decomposition, every matrix $F \in \mathrm{GL}^+(3)$ belongs to the G-orbit of some diagonal configuration D by some element $(e; A, B) \in G$. Hence, by Remark 3.1, the qualitative properties of a relative equilibrium $(D, (\xi_L, \xi_R))$ are the same as those of $(ADB^T, (A\xi_L, B\xi_R))$.

Let $J(k, r)$, with $k, r \in \mathbb{N}$ be the following family of integrals:

$$J(k, r) := \int_0^\infty \frac{s^r\, ds}{\Delta(F)^k},$$

and denote by V_i $(i = 1, 2)$ the partial derivative of V with respect to I_i and by V_{ij} the partial derivative of V_i with respect to I_j for $j = 1, 2$. Using (6), elementary calculus computations give:

$$V = -RJ(1, 0), \qquad V_1 = \frac{R}{2}J(3, 2), \qquad V_2 = \frac{R}{2}J(3, 1), \tag{16}$$

$$V_{11} = -\frac{3R}{4}J(5, 4), \qquad V_{12} = -\frac{3R}{4}J(5, 3), \qquad V_{22} = -\frac{3R}{4}J(5, 2). \tag{17}$$

Note that the integrals $J(k,r)$ are all positive as well as V_1 and V_2.

Next proposition gives the values of $J(k,r)$ in the case of spheroidal (two equal axes) configurations.

Proposition 4.1 *Let $F = \mathrm{diag}(a,a,c)$ be a spheroidal configuration with $a^2c = 1$.*

(i) *The integral $J(k,r)$ for the oblate spheroid F ($a > c$) and eccentricity $e = \sqrt{1-(\frac{c}{a})^2}$ is given by*

$$J_O(k,r) = \frac{2}{(1-e^2)^{\frac{2(r+1)-3k}{6}}} \int_0^1 \frac{(1-x^2)^r x^{3(k-1)-2r}\,dx}{(1-e^2x^2)^{\frac{k}{2}}}. \tag{18}$$

(ii) *The integral $J(k,r)$ for the prolate spheroid F ($a < c$) and eccentricity $e = \sqrt{1-(\frac{a}{c})^2}$ is given by*

$$J_P(k,r) = \frac{2}{(1-e^2)^{\frac{2(r+1)-3k}{3}}} \int_0^1 \frac{(1-x^2)^r x^{3(k-1)-2r}\,dx}{(1-e^2x^2)^{k}}. \tag{19}$$

Proof Note that for the diagonal configuration $F = \mathrm{diag}(a,b,c)$ the value of $\Delta(F)$ in definition (7) is

$$\Delta(F) = \left[\left(a^2+s\right)\left(b^2+s\right)\left(c^2+s\right)\right]^{1/2}.$$

For (i): making the change of variables $s = a^2\tan^2\theta$ we get

$$J_O(k,r) = 2a^{2(r+1)-3k} \int_0^{\pi/2} \frac{(\sin\theta)^{2r+1}(\cos\theta)^{3(k-1)-2r}}{(1+\frac{c^2-a^2}{a^2}\cos^2\theta)^{k/2}}\,d\theta.$$

Since the eccentricity is $e^2 = \frac{a^2-c^2}{a^2}$, then $a = (1-e^2)^{-1/6}$ and $c = (1-e^2)^{1/3}$ because $a^2c = 1$. Then from the above expression for $J_O(k,r)$, one gets

$$J_O(k,r) = \frac{2}{(1-e^2)^{\frac{2(r+1)-3k}{6}}} \int_0^{\pi/2} \frac{(\sin\theta)^{2r+1}(\cos\theta)^{3(k-1)-2r}}{(1-e^2\cos^2\theta)^{k/2}}\,d\theta.$$

Making $x = \cos\theta$ the result follows.

For (ii): The change of variables $s = c^2\tan^2\theta$ gives

$$J_P(k,r) = 2c^{2(r+1)-3k} \int_0^{\pi/2} \frac{(\sin\theta)^{2r+1}(\cos\theta)^{3(k-1)-2r}}{(1+\frac{a^2-c^2}{c^2}\cos^2\theta)^{k}}\,d\theta.$$

The eccentricity e of the prolate spheroid is such that $c^2(1-e^2) = a^2$. As $a^2c = 1$, then $a = (1-e^2)^{1/6}$ and $c = (1-e^2)^{-1/3}$ and the result follows for $x = \cos\theta$. □

As stated in the previous section, in order to find critical points of a G-invariant function defined in SL(3) we will work with its extension to $\mathrm{GL}^+(3)$ subjected to the constraint $\det F = 1$. Since any such function can be written as $f(F) = f(I_1(F), I_2(F))$, in order to compute the critical points we use the augmented function $f^\lambda(F) = f(I_1(F), I_2(F)) - \lambda \det(F)$ subjected to the condition $\det(F) = 1$. For the differentiation of f^λ consider the pairing between vectors $\delta F \in T_F\mathrm{SL}(3)$ and covectors $B \in T_F^*\mathrm{SL}(3)$ to be $B \cdot \delta F = \mathrm{tr}(B^T \delta F)$. Then using the chain rule we get that critical points must verify the following set of equations:

$$\begin{aligned} &\mathbf{d} f^\lambda(F) \cdot \delta F = 2\,\mathrm{tr}\Big[\Big(f_1 F^T + f_2\big(I_1 F^T - F^T F F^T\big) - \frac{\lambda}{2}\det(F) F^{-1}\Big)\delta F\Big] = 0, \\ &\qquad \det(F) = 1, \end{aligned} \tag{20}$$

since

$$\mathbf{d} I_1(F) \cdot \delta F = 2\,\mathrm{tr}\big(F^T \delta F\big), \tag{21}$$

$$\mathbf{d} I_2(F) \cdot \delta F = 2\big[I_1\,\mathrm{tr}\big(F^T \delta F\big) - \mathrm{tr}\big(F^T F F^T \delta F\big)\big], \tag{22}$$

$$\mathbf{d}\det(F) \cdot \delta F = \det(F)\mathrm{tr}\big(F^{-1}\delta F\big) \tag{23}$$

(see, for instance Ciarlet 1988 or Marsden and Hughes 1983).

Next proposition gives the expression for the locked inertia tensor.

Proposition 4.2 *The locked inertia tensor for the* G*-action on* $T^*\mathrm{GL}^+(3)$, *at a configuration* $F \in \mathrm{GL}^+(3)$, *is defined by*

$$\begin{aligned} &\big\langle\big(\widehat{\xi_L}, \widehat{\xi_R}\big), \mathbb{I}(F)\big(\widehat{\eta_L}, \widehat{\eta_R}\big)\big\rangle \\ &\quad = T\,\mathrm{tr}\big[\widehat{\xi_L}^T \widehat{\eta_L} F F^T + \widehat{\xi_R}^T F^T F \widehat{\eta_R} - \widehat{\xi_L}^T F \widehat{\eta_R} F^T - \widehat{\xi_R}^T F^T \widehat{\eta_L} F\big], \end{aligned} \tag{24}$$

where $T = \frac{4\pi}{15}\rho$ *and* $\widehat{\xi_i}, \widehat{\eta_i} \in \mathfrak{so}(3)$ *for* $i = 1, 2$.

Under the isomorphism (2), *the locked inertia tensor is also equivalent to*:

$$\big\langle(\xi_L, \xi_R), \mathbb{I}(F)(\eta_L, \eta_R)\big\rangle = T\begin{bmatrix}\xi_L & \xi_R\end{bmatrix}\begin{bmatrix}\mathbf{i}_S & -2\det(F)F^{-T} \\ -2\det(F)F^{-1} & \mathbf{i}_C\end{bmatrix}\begin{bmatrix}\eta_L \\ \eta_R\end{bmatrix}, \tag{25}$$

where $S = F F^T$, $C = F^T F$ *and* $\mathbf{i}_A = \mathrm{tr}(A)\mathbf{I} - A$ ($\mathbf{I}$ *denotes the identity matrix*).

Proof By the locked inertia tensor definition in Proposition 2.1, the expression (8) for the infinitesimal generators of the G-action on $\mathrm{GL}^+(3)$ and the definition (1) for the Riemannian metric, we have

$$\begin{aligned} \big\langle\big(\widehat{\xi_L}, \widehat{\xi_R}\big), \mathbb{I}(F)\big(\widehat{\eta_L}, \widehat{\eta_R}\big)\big\rangle &= \big\langle\!\big\langle\big(\widehat{\xi_L}, \widehat{\xi_R}\big)_{\mathrm{GL}^+(3)}(F), \big(\widehat{\eta_L}, \widehat{\eta_R}\big)_{\mathrm{GL}^+(3)}(F)\big\rangle\!\big\rangle \\ &= \big\langle\!\big\langle\widehat{\xi_L} F - F\widehat{\xi_R}, \widehat{\eta_L} F - F\widehat{\eta_R}\big\rangle\!\big\rangle \\ &= T\,\mathrm{tr}\big[\big(\widehat{\xi_L} F - F\widehat{\xi_R}\big)^T\big(\widehat{\eta_L} F - F\widehat{\eta_R}\big)\big]. \end{aligned}$$

Using the fact that $\widehat{\xi_i}$ and $\widehat{\eta_i}$ are skew-symmetric matrices and the cyclic property of the trace of a matrix, it is straightforward to obtain expression (24).

For the expression (25), we need some standard properties of the isomorphism (2). In particular,

$$\operatorname{tr}\bigl(\widehat{\xi}^T\widehat{\eta}\bigr) = 2\xi\cdot\eta, \tag{26}$$

$$\operatorname{tr}\bigl(\widehat{\xi}L\bigr) = \frac{1}{2}\operatorname{tr}\bigl(\widehat{\xi}\bigl(L-L^T\bigr)\bigr), \tag{27}$$

$$L\widehat{\xi}+\widehat{\xi}L = \widehat{\mathbf{i}_L v} \quad \text{if } L \text{ is a symmetric matrix}, \tag{28}$$

$$\widehat{L\xi} = \det(L)L^{-T}\widehat{\xi}L^{-1} \quad \text{if } L \text{ is an invertible matrix}, \tag{29}$$

where the dot denotes the standard inner product on $\mathbb{R}^3$. Let us compute some terms of the expression (24) since the other are done similarly

$$\begin{aligned}
\operatorname{tr}\bigl(\widehat{\xi_L}^T\widehat{\eta_L}FF^T\bigr) &= \frac{1}{2}\operatorname{tr}\bigl[\widehat{\xi_L}^T\bigl(\widehat{\eta_L}FF^T+FF^T\widehat{\eta_L}\bigr)\bigr] \quad (\text{by (27)})\\
&= \frac{1}{2}\operatorname{tr}\bigl(\widehat{\xi_L}^T\widehat{\mathbf{i}_S\eta_L}\bigr) \quad (\text{by (28)})\\
&= \xi_L\cdot\mathbf{i}_S\eta_L \quad (\text{by (26)}),\\
\operatorname{tr}\bigl(\widehat{\xi_R}^T F^T\widehat{\eta_L}F\bigr) &= \frac{1}{\det(F^{-1})}\operatorname{tr}\bigl[\widehat{\xi_R}^T\widehat{F^{-1}\eta_L}\bigr] \quad (\text{by (29)})\\
&= 2\det(F)\xi_R\cdot F^{-1}\eta_L \quad (\text{by (26)}).
\end{aligned}$$

□

As a straightforward consequence, we can obtain the momentum of a relative equilibrium for Dirichlet's model, that is, its angular momentum and circulation.

Corollary 4.1 *The momentum for a relative equilibrium with configuration F and velocity-vorticity pair $(\xi_L,\xi_R)\in\mathbb{R}^3\oplus\mathbb{R}^3$ is*

$$\mu = \mathbb{I}(F)(\xi_L,\xi_R) = T\bigl(\mathbf{i}_S\xi_L - 2\det(F)F^{-T}\xi_R, \mathbf{i}_C\xi_R - 2\det(F)F^{-1}\xi_L\bigr). \tag{30}$$

That is, the angular momentum and circulation of a Riemann ellipsoid with configuration by F, and angular velocity-vorticity pair (ξ_L,ξ_R) are given, respectively, by

$$\begin{aligned}
\mathfrak{j}/T &= \mathbf{i}_S\xi_L - 2\det(F)F^{-T}\xi_R,\\
\mathfrak{c}/T &= \mathbf{i}_C\xi_R - 2\det(F)F^{-1}\xi_L.
\end{aligned}$$

The expression for the twice augmented potential $V^{\lambda}_{\xi_L,\xi_R}$ follows now easily from Proposition 4.2.

$$\begin{aligned}
V^{\lambda}_{\xi_L,\xi_R}(F) = -R\int_0^\infty \frac{ds}{\Delta(F)}\\
&- T\left(\frac{1}{2}\xi_L\cdot\mathbf{i}_S\xi_L + \frac{1}{2}\xi_R\cdot\mathbf{i}_C\xi_R - 2\det(F)\xi_L\cdot F^{-T}\xi_R\right)\\
&- \lambda\det(F).
\end{aligned} \tag{31}$$

5 Existence Conditions for Symmetric Riemann Ellipsoids

In this section, we classify symmetric relative equilibria for Dirichlet's model. We will treat the spherical case (i.e., a configuration having three equal principal axes) as a particular case of a symmetric configuration. From the singular value decomposition and the definition of the action of $G = \mathbb{Z}_2 \ltimes (\mathrm{SO}(3) \times \mathrm{SO}(3))$ on $M = \mathrm{GL}^+(3)$ (or on $\mathrm{SL}^+(3)$), it follows that the stabilizer of a symmetric configuration F is conjugate to the stabilizer of a diagonal configuration. That is, conjugate to $\mathbb{Z}_2 \ltimes \mathrm{O}(2)^D_{\mathbf{e}}$ or $\mathbb{Z}_2 \ltimes \mathrm{SO}(3)^D$ if F has 2 or 1 different singular values, respectively (see Roberts and Sousa-Dias 1999 for a derivation of this result). Actually, if the configurations are diagonal, these are exactly their stabilizers. Here, K^D denotes the diagonal embedding of $K \subset \mathrm{SO}(3)$ in $\mathrm{SO}(3) \times \mathrm{SO}(3)$ and $\mathrm{O}(2)_{\mathbf{e}}$ is the subgroup of $\mathrm{SO}(3)$ generated by all the rotations $R_\theta \in \mathrm{SO}(2)_{\mathbf{e}}$ around a given axis $\mathbf{e}$ in $\mathbb{R}^3$ and a rotation, $\Pi_{\mathbf{e}^\perp}$, by π around an axis $\mathbf{e}^\perp$ perpendicular to $\mathbf{e}$. In case of the diagonal configuration $F = \mathrm{diag}(a, a, c)$, R_θ is the rotation matrix by an angle θ around $(0, 0, 1)$ and $\Pi_{\mathbf{e}^\perp}$ can be chosen to be $\mathrm{diag}(1, -1, -1)$. We introduce the following subgroups:

- $\mathrm{SO}(2)_{\mathbf{e}} \,\widetilde{\times}\, \mathrm{SO}(2)_{\mathbf{e}}$, generated by elements $(e; R_{\theta_1}, R_{\theta_2})$, with $R_{\theta_{1,2}} \in \mathrm{SO}(2)_{\mathbf{e}}$ and $(\sigma; \Pi_{\mathbf{e}^\perp}, \Pi_{\mathbf{e}^\perp})$.
- $\widetilde{\mathrm{O}(2)}_{\mathbf{e}}$, generated by elements $(e; R_\theta, R_\theta)$, with $R_\theta \in \mathrm{SO}(2)_{\mathbf{e}}$ and $(\sigma; \Pi_{\mathbf{e}^\perp}, \Pi_{\mathbf{e}^\perp})$.
- $\mathbb{Z}_2(\mathbf{e})$, the cyclic group isomorphic to $\mathbb{Z}_2$ generated by the element $(e; \Pi_{\mathbf{e}}, \Pi_{\mathbf{e}})$.
- More generally, if K is a subgroup of $\mathrm{SO}(3) \times \mathrm{SO}(3)$, we denote also by K the subgroup of $\mathbb{Z}_2 \ltimes (\mathrm{SO}(3) \times \mathrm{SO}(3))$ generated by elements $(e; k)$, with $k \in K$.

Note that since we are going to impose the constraint $F \in \mathrm{SL}(3)$, we will consider only two kinds of symmetric configurations, specifically:

- Spherical: $F = \mathrm{diag}(1, 1, 1)$.
- Spheroidal: $F = \mathrm{diag}(a, a, c)$, with $a^2 c = 1$.

To find all the possible Riemann ellipsoids with symmetric configurations, we will have to solve (15) with F of the above forms and different pairs (ξ_L, ξ_R). The possible solutions are summarized in the following theorem.

Theorem 5.1 *The relative equilibria for Dirichlet's model of a self-gravitating fluid mass are*:

(i) *The spherical equilibrium with spherical configuration* $F = \mathrm{diag}(1, 1, 1)$, *velocity-vorticity pair* $(0, 0)$ *and Lagrange multiplier* $\lambda = 2V_1 + 4V_2$. *Its corresponding momentum and isotropy groups are*

$$\mu = (\mathbf{j}, \mathbf{c}) = (0, 0), \quad G_\mu = \mathbb{Z}_2 \ltimes \big(\mathrm{SO}(3) \times \mathrm{SO}(3)\big),$$
$$G_F = G_{p_F} = \mathbb{Z}_2 \ltimes \mathrm{SO}(3)^D.$$

(ii) *The family of MacLaurin spheroids which have oblate spheroidal configurations* $F = \mathrm{diag}(a, a, c)$ (*with* $c < a$) *and angular velocity and vorticity parallel to the axis of symmetry* $\mathbf{e}_3$. *In terms of the parameter* Ω *defined by* $\Omega \mathbf{e}_3 = \xi_L - \xi_R$, *this family is characterized by* $\lambda = 2(1-e^2)^{2/3}V_1 + 4(1-e^2)^{1/3}V_2$ *and the following constraint between* Ω *and the eccentricity* e:

$$\frac{\Omega^2}{\pi\rho G} = 2\frac{\sqrt{1-e^2}}{e^3}\big(3-2e^2\big)\arcsin e - \frac{6}{e^2}\big(1-e^2\big). \tag{32}$$

Its corresponding momentum and isotropy groups are:

$$\mu = (\mathbf{j},\mathbf{c}) = 2\big(1-e^2\big)^{-1/3}T\Omega(\mathbf{e_3}, -\mathbf{e_3}),$$
$$G_F = \mathbb{Z}_2 \ltimes \mathrm{O}(2)^D_{\mathbf{e_3}}, \qquad G_\mu = \widetilde{\mathrm{SO}(2)_{\mathbf{e_3}} \times \mathrm{SO}(2)_{\mathbf{e_3}}}, \qquad G_{p_F} = \widetilde{\mathrm{O}(2)_{\mathbf{e_3}}}.$$

(iii) *Two branches of transversal spheroids which have prolate spheroidal configurations* $F = \mathrm{diag}(a,a,c)$ (*with* $c > a$). *We distinguish the two branches of this family with the signs* $+$ *and* $-$. *These branches are characterized by the Lagrange multiplier* $\lambda = 2((1-e^2)^{1/3}V_1 + (1-e^2)^{-1/3}(e^2-2)V_2)$, *the velocity-vorticity pair* $(\xi_L, \xi_R)_\pm = \omega_\pm(\mathbf{n}, f_\pm\mathbf{n})$ *with* $f_\pm = \frac{1\pm e}{\sqrt{1-e^2}}$ (*where* $\mathbf{n}$ *is a unit vector perpendicular to* $\mathbf{e_3}$) *and the following constraints between* $\omega_\pm$ *and the eccentricity* e:

$$\frac{\omega_\pm^2}{\pi\rho G} = \mp\frac{(e\mp 1)^2(e\pm 1)}{e^5}\big(3e + \big(e^2-3\big)\operatorname{arctanh} e\big). \tag{33}$$

The corresponding momentum and isotropy groups are:

$$\mu_\pm = T\omega_\pm\bigg(-\frac{e(e\pm 2)}{(1-e^2)^{2/3}}\mathbf{n}, \pm\frac{e(e\mp 2)}{(e\mp 1)(1-e^2)^{1/6}}\mathbf{n}\bigg),$$
$$G_F = \mathbb{Z}_2 \ltimes \mathrm{O}(2)^D_{\mathbf{e_3}}, \qquad G_\mu = \mathrm{SO}(2)_{\mathbf{n}} \times \mathrm{SO}(2)_{\mathbf{n}}, \qquad G_{p_F} = \mathbb{Z}_2(\mathbf{n}).$$

Before proving the theorem, we remark that formula (32) has already been obtained by MacLaurin in 1742, as it is claimed in p. 4 of Chandrasekhar's book (Chandrasekhar 1987).

Proof First, using (31), (16), and (20), it is easy to see that the general conditions (15) are equivalent to

$$\begin{aligned}
0 &= 2V_1F^T + 2V_2(I_1F^T - F^TFF^T) - \lambda\det(F)F^{-1} \\
&\quad - T\big[\big(\|\xi_L\|^2 + \|\xi_R\|^2\big)F^T - F^T(\xi_L\otimes\xi_L) - (\xi_R\otimes\xi_R)F^T \\
&\quad + 2\det(F)\big(\big(F^{-1}\xi_L\otimes F^{-T}\xi_R\big) - \big(\xi_L\cdot F^{-T}\xi_R\big)F^{-1}\big)\big],
\end{aligned} \tag{34}$$
$$1 = \det(F). \tag{35}$$

Spherical case: If $F = \mathbf{I}$, then (34), (35) are simply

$$\begin{aligned}
&\big(2V_1 + 4V_2 - \lambda - T\big[\|\xi_L\|^2 + \|\xi_R\|^2 - 2\xi_L\cdot\xi_R\big]\big)\mathbf{I} \\
&\quad + T[\xi_L\otimes\xi_L + \xi_R\otimes\xi_R - 2\xi_L\otimes\xi_R] = 0.
\end{aligned}$$

Taking $\xi_L = (\xi_{L,1}, \xi_{L,2}, \xi_{L,3})$ and the same sort of notation for ξ_R, the off-diagonal terms of this expression are independent of V_1, V_2, and λ, and equivalent to the following 6 equations:

$$
\begin{aligned}
&(\xi_{L,1}-\xi_{R,1})(\xi_{L,2}-\xi_{R,2})=0, \qquad && \xi_{R,1}\xi_{L,2}=\xi_{R,2}\xi_{L,1},\\
&(\xi_{L,1}-\xi_{R,1})(\xi_{L,3}-\xi_{R,3})=0, \qquad && \xi_{R,1}\xi_{L,3}=\xi_{R,3}\xi_{L,1},\\
&(\xi_{L,2}-\xi_{R,2})(\xi_{L,3}-\xi_{R,3})=0, \qquad && \xi_{R,2}\xi_{L,3}=\xi_{R,3}\xi_{L,2}.
\end{aligned}
$$

It follows then that $\xi_L = \xi_R$. Recall from Theorem 2.1 that the momentum of a relative equilibrium with configuration x and velocity ξ is given by $\mu = \mathbb{I}(x)(\xi)$. Then from (25), we have $\mu = (\mathbf{j}, \mathbf{c}) = (0, 0)$. Therefore, $G_F = \mathbb{Z}_2 \ltimes \mathrm{SO}(3)^D$, $G_\mu = G$, $G_{p_F} = G_\mu \cap G_F = G_F$.

Now, noting that $(\xi_L, \xi_R) \in \mathfrak{g}_{p_F}$ if $\xi_L = \xi_R$, and that the velocity of a relative equilibrium is defined only up to addition of elements in $\mathfrak{g}_{p_F}$, the relative equilibrium $(\mathbf{I}, (\xi_L, \xi_R))$ is the same as $(\mathbf{I}, (0, 0))$. Then the relative equilibrium conditions are satisfied with $\lambda = 2V_1 + 4V_2$.

Spheroidal case: We now consider $F = \mathrm{diag}(a, a, c)$ with $a^2c = 1$. Since in the $\langle \mathbf{e_1}, \mathbf{e_2} \rangle$ plane all directions are equivalent, we can assume without loss of generality that $\xi_{L,1} = 0$. Now, conditions (34), (35) are equivalent to the following system:

$$
\begin{aligned}
0 &= 2aV_1 + 2a^3V_2 + 2ac^2V_2 - ac\lambda - aT\big(\xi_{R,2}^2 + \xi_{R,3}^2 + \xi_{L,2}^2 + \xi_{L,3}^2\big)\\
&\quad + 2T(c\xi_{R,2}\xi_{L,2} + a\xi_{R,3}\xi_{L,3}), \qquad (36)\\
0 &= 2aV_1 + 2a^3V_2 + 2ac^2V_2 - ac\lambda - aT\big(\xi_{R,1}^2 + \xi_{R,3}^2 + \xi_{L,3}^2\big)\\
&\quad + 2aT\xi_{R,3}\xi_{L,3}, \qquad (37)\\
0 &= 2cV_1 + 4a^2cV_2 - a^2\lambda - cT\big(\xi_{R,1}^2 + \xi_{R,2}^2 + \xi_{L,2}^2\big) + 2aT\xi_{R,2}\xi_{L,2}, \qquad (38)\\
0 &= \xi_{R,1}\xi_{R,2} = \xi_{R,1}\xi_{R,3} = \xi_{R,1}\xi_{L,2} = \xi_{R,1}\xi_{L,3}, \qquad (39)\\
0 &= cT\xi_{R,2}\xi_{R,3} - 2aT\xi_{R,3}\xi_{L,2} + aT\xi_{L,2}\xi_{L,3}, \qquad (40)\\
0 &= aT\xi_{R,2}\xi_{R,3} - 2aT\xi_{R,2}\xi_{L,3} + cT\xi_{L,2}\xi_{L,3}, \qquad (41)\\
1 &= a^2c. \qquad (42)
\end{aligned}
$$

Note that these equations imply that $\xi_{R,1} = 0$. Indeed, if $\xi_{R,1} \neq 0$, then (39) implies $\xi_{R,2} = \xi_{R,3} = \xi_{L,2} = \xi_{L,3} = 0$ and so (36) and (37) imply $\xi_{R,1} = 0$ which is a contradiction. We will now proceed systematically considering four main cases: (i) $\xi_L = \xi_R = 0$, (ii) $\xi_{L,2} = 0$, (iii) $\xi_{L,3} = 0$ and $\xi_{L,2} \neq 0$, and (iv) $\xi_{L,3} \neq 0$ and $\xi_{L,2} \neq 0$.

(i) If $\xi_L = \xi_R = 0$, then (37), (38), and (42) are the only nontrivial conditions, which are equivalent to

$$
a^2c = 1, \qquad \lambda = \frac{c}{a^2}\big(2V_1 + 4a^2V_2\big), \qquad \big(a^6 - 1\big)\big(V_1 + a^2V_2\big) = 0.
$$

As V_1 and V_2 are positive and $0 < a \neq 1$ it follows from the last equation that there is no solution.

(ii) If $\xi_{L,2} = 0$, then from (36) and (37) we have $\xi_{R,2} = 0$ and so (38) and (42) give $\lambda = 2c^2 V_1 + 4c V_2$.

Let $\Omega = \xi_{L,3} - \xi_{R,3}$. Then the remaining nontrivial equations, (36), (37), and (42), give

$$\Omega^2 = \frac{2}{T}\frac{a^2 - c^2}{a^2}\bigl(V_1 + a^2 V_2\bigr). \tag{43}$$

As V_1 and V_2 are positive, then last equality implies that the spheroidal configuration F is oblate, that is $a > c$. The eccentricity of the spheroid is $e^2 = 1 - \frac{c^2}{a^2}$ and $a^2 = (1 - e^2)^{-1/3}$. Using the relations (16) for the partial derivatives of the self-gravitating potential, V_1 and V_2, the expression J_O for the integrals $J(k, r)$ in the oblate case given by (18) and $\frac{R}{T} = 2\pi\rho G$, then (43) is equivalent to

$$\begin{aligned}\Omega^2 &= \frac{R}{T}e^2\bigl(J_O(3,2) + \bigl(1 - e^2\bigr)^{-1/3} J_O(3,1)\bigr)\\ &= 2\pi\rho G e^2\bigl(J_O(3,2) + \bigl(1 - e^2\bigr)^{-1/3} J_O(3,1)\bigr).\end{aligned}$$

One can easily compute the definite integrals $J_O(3, 2)$ and $J_O(3, 1)$, although we avoid to display their expressions since they are quite lengthy. However, the expression $J_O(3, 2) + (1 - e^2)^{-1/3} J_O(3, 1)$ is

$$J_O(3,2) + \bigl(1 - e^2\bigr)^{-1/3} J_O(3,1) = 3\frac{e^2 - 1}{e^4} + \frac{3 - 2e^2}{e^5}\sqrt{1 - e^2}\arcsin e,$$

from which (32) follows. From Corollary 4.1, it is trivial to obtain that the momentum of this relative equilibrium. Hence, using (9), the appropriate isotropy groups are also straightforward.

(iii) If $\xi_{L,3} = 0$ and $\xi_{L,2} \neq 0$, it follows from (40) and (41) that $\xi_{R,3} = 0$. Then (37) and (42) give

$$a^2 c = 1, \qquad \lambda = 2a^2\bigl(V_1 + \bigl(a^2 + c^2\bigr)V_2\bigr).$$

We can set $\mathbf{n} = \mathbf{e_2}$ and $(\xi_L, \xi_R) = \omega(\mathbf{n}, f\mathbf{n})$. So, substituting the above value of λ into (36) and (38) these equations are

$$0 = \bigl(a - 2cf + af^2\bigr)T\omega^2, \tag{44}$$

$$0 = 2\bigl(c^2 - a^2\bigr)V_1 + 2a^2\bigl(c^2 - a^2\bigr)V_2 - \bigl(c^2 f^2 - 2acf + c^2\bigr)T\omega^2. \tag{45}$$

From (44), we obtain the solutions $f_\pm = \frac{c \pm \sqrt{c^2 - a^2}}{a}$, from which follows that the spheroids must be prolate ($c > a$). In terms of the eccentricity $e^2 = 1 - \frac{a^2}{c^2}$, we have

$$f_\pm = \frac{1 \pm e}{\sqrt{1 - e^2}}.$$

Therefore, (45) gives

$$\omega_{\pm}^2 = \frac{2(c^2 - a^2)(V_1 + a^2 V_2)}{T(c^2 f_{\pm}^2 - 2acf_{\pm} + c^2)} = \frac{1 \mp e}{T}\big(V_1 + \big(1 - e^2\big)^{1/3} V_2\big).$$

Substituting in this expression $V_1 = \frac{R}{2} J_P(3,2)$ and $V_2 = \frac{R}{2} J_P(3,1)$, as well as $R = 2\pi\rho G T$ gives (33). As before, using the expression of the locked inertia tensor and (9) the remaining results follow.

(iv) In this case, we have $\xi_{L,2} \neq 0$, $\xi_{L,3} \neq 0$ and $\xi_{L,1} = \xi_{R,1} = 0$. Note that from (40) and (41) one should also have $\xi_{R,2} \neq 0$ and $\xi_{R,3} \neq 0$. So, let $\xi_{L,2} = g\xi_{R,2}$ and $\xi_{L,3} = h\xi_{R,3}$ for some reals $g, h \neq 0$. Then using also (42), (40) and (41) reduce to

$$\big(1 - 2a^3 g + a^3 gh\big)\xi_{R,2}\xi_{R,3} = 0,$$
$$\big(a^3 - 2a^3 h + gh\big)\xi_{R,2}\xi_{R,3} = 0.$$

These equations have solutions

$$(h_{\pm}, g_{\pm}) = \left(\frac{5a^6 - 1 \pm \sqrt{1 - 10a^6 + 9a^{12}}}{4a^6}, \frac{1 + 3a^6 \pm \sqrt{1 - 10a^6 + 9a^{12}}}{4a^3}\right). \quad (46)$$

Comparing (36) and (37) using $\xi_{L,2} = g\xi_{R,2}$, $\xi_{L,3} = h\xi_{R,3}$ and (42), we obtain $-a(1 + g^2) + 2cg = 0$ and so $g_{\pm}$ must be $\frac{1+\sqrt{1-a^6}}{a^3}$ or $\frac{1-\sqrt{1-a^6}}{a^3}$, but this is a contradiction with (46). So, there is no solution for the above system. □

Remark 5.1 Note that the functions f_+ and f_- appearing in the characterization of transversal spheroids are inverse of each other. Therefore, the two families of transversal spheroids belong to a single orbit of the symmetry group. Indeed, the $\mathbb{Z}_2$ symmetry interchanges the $+$ and $-$ families, since $\sigma \cdot (F; \omega_+ \mathbf{n}, \omega_+ f_+ \mathbf{n}) = (F^T; \omega_+ f_+ \mathbf{n}, \omega_+ \mathbf{n}) = (F; \omega_- \mathbf{n}, \omega_- f_- \mathbf{n})$, as it follows from their definitions that $\omega_+^2/\omega_-^2 = f_-/f_+$.

Remark 5.2 Theorem 5.1 is in agreement with Riemann's classification of ellipsoidal figures of equilibrium for Dirichlet's model of self-gravitating fluid masses. That is, these ellipsoidal figures of equilibrium must lie in one of the following categories: (a) the case of a uniform rotation with no internal motion (or uniform vorticity and no rotation); (b) the case when the directions of the angular velocity ξ_L and vorticity ξ_R are the same and coincide with a principal axis of the ellipsoid (also known as ellipsoids of type S); (c) The case when the angular velocity and vorticity are not parallel, but lie in the same principal plane.

In particular, we show that for Dirichlet's model it is not possible to obtain relative equilibria with spheroidal configurations belonging to category (c).

Remark 5.3 The existence of transversal spheroids is referred in Chandrasekhar's book (Chandrasekhar 1987) (see, for instance p. 143); however, their study is not present in the classical works of Liapunov (1904) and Poincaré (1885).

In the work of Chandrasekhar, the transversal spheroids have been treated as particular (or limiting) cases of some families of triaxial ellipsoids of type S. We feel that the transversal spheroids should be regarded as independent solutions in their own right due to the fundamental geometric differences (different isotropies) from ellipsoids with three distinct length axes.

Our expression for the angular velocity of transversal spheroids coincides with (93) in Chap. 7 of Chandrasekhar (1987). Our expression for $f_\pm$ coincides with equation (92) in the same page of the referred book once we substitute the vorticity ξ_R by

$$\xi_R' = -\frac{a^2+c^2}{ac}\xi_R,$$

which is the vorticity in the reference frame rotating with the body. In this accelerated frame (the one considered by Chandrasekhar), the spheroid is at rest and there are only internal motions with constant vorticity ξ_R'. Therefore, ξ_R' is the vorticity vector expressed in body coordinates. See Remark 3 in Sect. 3 of Roberts and Sousa-Dias (1999) for a more detailed explanation of these differences.

6 Stability Conditions for Symmetric Riemann Ellipsoids

In this section, we apply the singular version of the reduced energy-momentum method introduced in Rodríguez-Olmos (2006) in order to deduce the stability of the symmetric relative equilibria obtained in Theorem 5.1. In order to apply this method, it is essential to compute the second derivative of the twice augmented potential $V^\lambda_{\xi_L,\xi_R}$. The following lemma gives that result.

Lemma 6.1 *If F is a critical point of the twice augmented potential*

$$V^\lambda_{(\xi_L,\xi_R)} = V\big(I_1(F), I_2(F)\big) - \frac{1}{2}\begin{bmatrix}\xi_L & \xi_R\end{bmatrix}\mathbb{I}(F)\begin{bmatrix}\xi_L \\ \xi_R\end{bmatrix} - \lambda\det(F),$$

for $(\xi_L,\xi_R)\in\mathbb{R}^3\times\mathbb{R}^3$ and $\mathbb{I}(F)$ as in Proposition 4.2 and $A, B\in T_F\mathrm{GL}^+(3)$, then

$$\begin{aligned}\mathbf{d}^2_F V^\lambda_{(\xi_L,\xi_R)}(A,B) = {}&\mathbf{d}^2_F V(A,B) - \frac{1}{2}\begin{bmatrix}\xi_L & \xi_R\end{bmatrix}\big(\mathbf{D}^2_F\mathbb{I}(A,B)\big)\begin{bmatrix}\xi_L \\ \xi_R\end{bmatrix}\\ &- \lambda\det(F)\big(\operatorname{tr}\big(F^{-1}B\big)\operatorname{tr}\big(F^{-1}A\big) - \operatorname{tr}\big(F^{-1}BF^{-1}A\big)\big),\end{aligned}$$

where

$$\begin{aligned}\mathbf{d}^2_F V(A,B) = {}&2\operatorname{tr}\big(B^TA\big)(V_1+I_1V_2)\\ &-2\operatorname{tr}\big(BF^TFA^T + FB^TFA^T + FF^TBA^T\big)V_2\\ &+4\operatorname{tr}\big(F^TA\big)\operatorname{tr}\big(FB^T\big)\big(V_2+V_{11}+2I_1V_{12}+I_1^2V_{22}\big)\\ &-4\operatorname{tr}\big(FF^TFB^T\big)\operatorname{tr}\big(F^TA\big)(V_{12}+I_1V_{22})\end{aligned}$$

$$-4\,\mathrm{tr}\big(F^T F F^T A\big)\,\mathrm{tr}\big(F B^T\big)(V_{12}+I_1 V_{22})$$
$$+4\,\mathrm{tr}\big(F^T F F^T A\big)\,\mathrm{tr}\big(F F^T F B^T\big)V_{22}$$

and

$$\begin{bmatrix}\xi_L & \xi_R\end{bmatrix}\mathbf{D}^2_F\mathbb{I}(A,B)\begin{bmatrix}\xi_L\\ \xi_R\end{bmatrix}=T\,\mathrm{tr}\big(4\widehat{\xi_R}B^T\widehat{\xi_L}A-2\widehat{\xi_L}^2AB^T-2\widehat{\xi_R}^2B^TA\big).$$

Proof We will just sketch the computation of $\mathbf{d}^2_F V(A,B)$.

Recall from the proof of Theorem 5.1, that

$$\mathbf{d}V(F)\cdot A=2(V_1+I_1V_2)\,\mathrm{tr}\big(F^TA\big)-2V_2\,\mathrm{tr}\big(F^TFF^TA\big).$$

Differentiating again using the expressions for $\mathbf{d}I_1(F)\cdot A$ and $\mathbf{d}I_2(F)\cdot A$ given in (21) and (22) and the chain rule, the result follows.

For the expression $\mathbf{D}^2_F\mathbb{I}(A,B)$, we differentiate the expression (24), which in this case takes the form

$$\big\langle\big(\widehat{\xi_L},\widehat{\xi_R}\big),\mathbb{I}(F)\big(\widehat{\xi_L},\widehat{\xi_R}\big)\big\rangle=T\,\mathrm{tr}\big[2\widehat{\xi_R}F^T\widehat{\xi_L}F-F^TF\widehat{\xi_R}^2-FF^T\widehat{\xi_L}^2\big].$$

Then applying standard properties of the trace, we get

$$\big\langle\big(\widehat{\xi_L},\widehat{\xi_R}\big),(\mathbf{D}\mathbb{I}(F)\cdot A)\big(\widehat{\xi_L},\widehat{\xi_R}\big)\big\rangle=T\,\mathrm{tr}\big[4\widehat{\xi_R}A^T\widehat{\xi_L}F-2A^TF\widehat{\xi_R}^2-2AF^T\widehat{\xi_L}^2\big].$$

The expression for $\mathbf{D}^2_F\mathbb{I}(A,B)$ stated follows now easily. Finally, using (23) to differentiate the expression

$$\mathbf{d}\det(F)\cdot A=\det(F)\,\mathrm{tr}\big(F^{-1}A\big),$$

we get

$$\mathbf{d}^2_F\big(\det(F)\big)(A,B)=\det(F)\big[\mathrm{tr}(F^{-1}B)\,\mathrm{tr}\big(F^{-1}A\big)-\mathrm{tr}\big(F^{-1}BF^{-1}A\big)\big].\qquad\square$$

6.1 Spherical Equilibrium

We now study the stability of the spherical equilibrium. Notice from the outline of the method in Sect. 3 that for this equilibrium we have that $\mathfrak{q}^\mu$, the correction term and the Arnold form are all trivial, as well as the velocity-vorticity pair (ξ_L,ξ_R). As a consequence, $\Sigma^{\mathrm{SL}(3)}_{\mathrm{int}}=\mathbf{S}^{\mathrm{SL}(3)}$, the orthogonal complement to the G-orbit at the identity in SL(3). Hence, to conclude stability of the spherical equilibrium we need to study the definiteness of

$$\mathbf{d}^2_{\mathbf{I}}V^\lambda_{(0,0)}\Big|_{\mathbf{S}^{\mathrm{SL}(3)}}.$$

Theorem 6.1 *For Dirichlet's model, the spherical equilibrium is nonlinearly G-stable.*

Proof Recall that $T_{\mathbf{I}}\mathrm{SL}(3)$ is the space of traceless matrices. Also, the infinitesimal action of $\mathfrak{g}$ on $\mathrm{GL}^+(3)$ at $\mathbf{I}$ is $(\xi_L, \xi_R)_{\mathrm{GL}^+(3)}(\mathbf{I}) = \widehat{\xi_L} - \widehat{\xi_R}$. Then

$$\mathbf{S}^{\mathrm{SL}(3)} = \left\{A \in T_{\mathbf{I}}\mathrm{SL}(3) : \mathrm{tr}\left(A\widehat{\xi_L} - A\widehat{\xi_R}\right) = 0 \; \forall \xi_L, \xi_R \in \mathbb{R}^3\right\}.$$

Therefore, $\mathbf{S}^{\mathrm{SL}(3)}$ is the space of traceless symmetric matrices. That is, matrices of the form

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & -(a_{11}+a_{22}) \end{bmatrix}.$$

We fix a basis for $\mathbf{S}^{\mathrm{SL}(3)}$ with respect to which the components of A are $(a_{11}, a_{22}, a_{12}, a_{13}, a_{23})$.

By Lemma 6.1, the expression of $\mathbf{d}^2_{\mathbf{I}} V^{\lambda}_{(0,0)}|_{\mathbf{S}^{\mathrm{SL}(3)}}$, with $\lambda = 2V_1 + 4V_2$ as given in Theorem 5.1, reduces to:

$$\mathbf{d}^2_{\mathbf{I}} V^{\lambda}_{(0,0)}\big|_{\mathbf{S}^{\mathrm{SL}(3)}}(A, B) = 4(V_1 + V_2)\,\mathrm{tr}(BA),$$

where we have applied the fact that A and B are traceless symmetric matrices. Therefore, in this basis, we have

$$\mathbf{d}^2_{\mathbf{I}} V^{\lambda}_{(0,0)}\big|_{\mathbf{S}^{\mathrm{SL}(3)}} = 4(V_1 + V_2)\begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}.$$

As $V_1 + V_2$ is positive, the eigenvalues of this matrix are $12(V_1 + V_2), 4(V_1 + V_2)$ and $8(V_1 + V_2)$ with multiplicities 1, 1 and 3, respectively. These are all positive, therefore, the spherical equilibrium is G-stable. □

6.2 MacLaurin Spheroids

We now study the nonlinear stability of MacLaurin spheroids in the setup of previous sections (Theorem 5.1). As it has been stated, a MacLaurin spheroid has an oblate configuration which, with no loss of generality, we suppose diagonal. This configuration is uniquely characterized by the eccentricity $e \in (0, 1)$. In order to apply Theorem 3.1, one needs first to split $\mathfrak{g} = \mathbb{R}^3 \oplus \mathbb{R}^3$ according to (10), that is, as

$$\mathfrak{g}_\mu = \mathfrak{g}_{p_F} \oplus \mathfrak{p} \quad \text{and} \quad \mathfrak{g} = \mathfrak{g}_F \oplus \mathfrak{p} \oplus \mathfrak{t}.$$

Recall that for the MacLaurin spheroid one has $G_F = \mathbb{Z}_2 \ltimes \mathrm{O}(2)^D$ and $G_\mu = \widetilde{\mathrm{SO}(2)_{\mathbf{e_3}} \times \mathrm{SO}(2)_{\mathbf{e_3}}}$. One can then choose the following ordered orthonormal bases (with respect to the Euclidean product in $\mathbb{R}^3 \oplus \mathbb{R}^3$) for each of the spaces of the above splitting: If we define $h=\frac{1}{\sqrt{2}}(\mathbf{e_3},\mathbf{e_3})$, $p=\frac{1}{\sqrt{2}}(\mathbf{e_3},-\mathbf{e_3})$, $t_1=(\mathbf{e_1},0)$, $t_2=(0,\mathbf{e_1})$, $t_3=(\mathbf{e_2},0)$, $t_4=(0,\mathbf{e_2})$, then

$$\begin{aligned}\mathfrak{g}_F&=\operatorname{span}\{h\},\\ \mathfrak{p}&=\operatorname{span}\{p\},\\ \mathfrak{t}&=\operatorname{span}\{t_1,t_2,t_3,t_4\}.\end{aligned}$$

It is straightforward to check that these subspaces are invariant for $G_{P_F}=\widetilde{\mathrm{O}(2)_{\mathbf{e_3}}}$. The orthogonal velocity $\xi^\perp$ for the MacLaurin relative equilibrium is the orthogonal projection of the velocity ξ onto $\mathfrak{p}$. Then

$$\begin{aligned}(\xi_L,\xi_R)^\perp&=\frac{1}{2}(\xi_{L,3}\mathbf{e_3},\xi_{R,3}\mathbf{e_3})\cdot(\mathbf{e_3},-\mathbf{e_3})(\mathbf{e_3},-\mathbf{e_3})\\ &=\frac{1}{2}(\xi_{L,3}-\xi_{R,3})(\mathbf{e_3},-\mathbf{e_3})=\frac{\Omega}{2}(\mathbf{e_3},-\mathbf{e_3})=\frac{\Omega}{\sqrt{2}}p,\end{aligned}\tag{47}$$

where Ω must satisfy (32). As already defined, $\widehat{\mathbb{I}}_0$ is the restriction of $\mathbb{I}$ to $\mathfrak{p}\oplus\mathfrak{t}$. The locked inertia matrix for the configuration $F=\operatorname{diag}(a,a,c)$ is according to (25),

$$\mathbb{I}(F)=T\begin{bmatrix}D_1 & D_2\\ D_2 & D_1\end{bmatrix},$$

where $D_1=\operatorname{diag}(a^2+c^2,a^2+c^2,2a^2)$ and $D_2=-\operatorname{diag}(2ac,2ac,2a^2)$. It is now straightforward to obtain the $\widehat{\mathbb{I}}_0$ matrix with respect to the basis (p,t_1,t_2,t_3,t_4). That is,

$$\widehat{\mathbb{I}}_0=T\begin{bmatrix}4a^2 & 0 & 0 & 0 & 0\\ 0 & a^2+c^2 & -2ac & 0 & 0\\ 0 & -2ac & a^2+c^2 & 0 & 0\\ 0 & 0 & 0 & a^2+c^2 & -2ac\\ 0 & 0 & 0 & -2ac & a^2+c^2\end{bmatrix}.$$

Or in terms of the eccentricity,

$$\widehat{\mathbb{I}}_0=T\begin{bmatrix}\frac{4}{\sqrt[3]{1-e^2}} & 0 & 0 & 0 & 0\\ 0 & \frac{2-e^2}{\sqrt[3]{1-e^2}} & -2\sqrt[6]{1-e^2} & 0 & 0\\ 0 & -2\sqrt[6]{1-e^2} & \frac{2-e^2}{\sqrt[3]{1-e^2}} & 0 & 0\\ 0 & 0 & 0 & \frac{2-e^2}{\sqrt[3]{1-e^2}} & -2\sqrt[6]{1-e^2}\\ 0 & 0 & 0 & -2\sqrt[6]{1-e^2} & \frac{2-e^2}{\sqrt[3]{1-e^2}}\end{bmatrix}.\tag{48}$$

We can use $\widehat{\mathbb{I}}_0$ and $(\xi_L, \xi_R)^\perp$ to compute the momentum of a MacLaurin spheroid. Indeed,

$$\mu = \widehat{\mathbb{I}}_0(\xi_L, \xi_R)^\perp = \frac{\Omega}{\sqrt{2}}\widehat{\mathbb{I}}_0(p) = \frac{2\sqrt{2}T\Omega}{(1-e^2)^{1/3}}p,$$

which is of course the same as the value obtained in Theorem 5.1 under the identification $\mathfrak{g} \simeq \mathfrak{g}^*$ induced by the Euclidean product in $\mathbb{R}^3 \oplus \mathbb{R}^3$.

In order to apply Theorem 3.1, we need to verify that the singular Arnold form is nondegenerate.

Proposition 6.1 *For a MacLaurin spheroid $\mathfrak{q}^\mu = \mathfrak{t}$ and the Arnold form, defined in* (14), *is positive definite for all eccentricities.*

Proof Recall that the Arnold form $\mathrm{Ar} : \mathfrak{q}^\mu \times \mathfrak{q}^\mu \to \mathbb{R}$ is defined by:

$$\mathrm{Ar}(\gamma_1, \gamma_2) = \langle \mathrm{ad}^*_{\gamma_1}\mu, \Lambda(F, \mu)(\gamma_2)\rangle,$$

where

$$\Lambda(F, \mu)(\gamma) = \widehat{\mathbb{I}}_0^{-1}\big(\mathrm{ad}^*_\gamma \mu\big) + \mathbb{P}_{\mathfrak{p}^*\oplus\mathfrak{t}^*}\big[\mathrm{ad}_\gamma\big(\widehat{\mathbb{I}}_0^{-1}\mu\big)\big].$$

First, we compute the space $\mathfrak{q}^\mu$. Notice the following relations for the adjoint representation of G:

$$\mathrm{ad}_{t_1} p = \frac{-1}{\sqrt{2}}t_3, \qquad \mathrm{ad}_{t_2} p = \frac{1}{\sqrt{2}}t_4, \qquad \mathrm{ad}_{t_3} p = \frac{1}{\sqrt{2}}t_1, \qquad \mathrm{ad}_{t_4} p = \frac{-1}{\sqrt{2}}t_2, \tag{49}$$

and

$$\begin{aligned} \mathrm{ad}_{t_1} t_2 &= \mathrm{ad}_{t_1} t_4 = \mathrm{ad}_{t_2} t_3 = \mathrm{ad}_{t_3} t_4 = 0, \\ \mathrm{ad}_{t_1} t_3 &= \frac{1}{\sqrt{2}}(h+p), \\ \mathrm{ad}_{t_2} t_4 &= \frac{1}{\sqrt{2}}(h-p). \end{aligned} \tag{50}$$

Also, recall that under our identification $\mathfrak{g} \simeq \mathfrak{g}^*$, we have $\mathrm{ad}^*_\gamma \rho = -\mathrm{ad}_\gamma \rho$, for $\gamma \in \mathfrak{g}$, $\rho \in \mathfrak{g}^*$, and where ρ in the right-hand side is identified with an element of $\mathfrak{g}$. Therefore, $\mathbb{P}_{\mathfrak{g}_F}(\mathrm{ad}^*_{t_i}\mu) = 0$ for $i = 1, 2, 3, 4$, hence $\mathfrak{q}^\mu = \mathfrak{t}$. As $\frac{\Omega}{\sqrt{2}}\widehat{\mathbb{I}}_0(p) = \mu$, then $\widehat{\mathbb{I}}_0^{-1}(\mu) = \frac{\Omega}{\sqrt{2}}p$. Then from (49), we obtain

$$\begin{aligned} \mathrm{ad}_{t_1}\big(\widehat{\mathbb{I}}_0^{-1}\mu\big) &= -\frac{\Omega}{2}t_3, \qquad & \mathrm{ad}_{t_2}\big(\widehat{\mathbb{I}}_0^{-1}\mu\big) &= \frac{\Omega}{2}t_4, \\ \mathrm{ad}_{t_3}\big(\widehat{\mathbb{I}}_0^{-1}\mu\big) &= \frac{\Omega}{2}t_1, \qquad & \mathrm{ad}_{t_4}\big(\widehat{\mathbb{I}}_0^{-1}\mu\big) &= -\frac{\Omega}{2}t_2. \end{aligned}$$

The inverse of the matrix (48) is not difficult to compute. Here, we just state the values of $\widehat{\mathbb{I}}_0^{-1}(\mathrm{ad}^*_w\mu) = -\widehat{\mathbb{I}}_0^{-1}(\mathrm{ad}_w\mu)$ on vectors w of the fixed basis:

$$\widehat{\mathbb{I}}_0^{-1}\big(\mathrm{ad}^*_{t_1}\mu\big) = \frac{2T\Omega}{(1-e^2)^{1/3}}\widehat{\mathbb{I}}_0^{-1}(t_3) = \frac{-2(e^2-2)\Omega}{e^4}t_3 + \frac{4\sqrt{1-e^2}\Omega}{e^4}t_4,$$

$$\widehat{\mathbb{I}}_0^{-1}\big(\mathrm{ad}^*_{t_2}\mu\big) = -\frac{2T\varOmega}{(1-e^2)^{1/3}}\widehat{\mathbb{I}}_0^{-1}(t_4) = \frac{-4\sqrt{1-e^2}\varOmega}{e^4}t_3 + \frac{2(e^2-2)\varOmega}{e^4}t_4,$$
$$\widehat{\mathbb{I}}_0^{-1}\big(\mathrm{ad}^*_{t_3}\mu\big) = -\frac{2T\varOmega}{(1-e^2)^{1/3}}\widehat{\mathbb{I}}_0^{-1}(t_1) = \frac{2(e^2-2)\varOmega}{e^4}t_1 - \frac{4\sqrt{1-e^2}\varOmega}{e^4}t_2,$$
$$\widehat{\mathbb{I}}_0^{-1}\big(\mathrm{ad}^*_{t_4}\mu\big) = \frac{2T\varOmega}{(1-e^2)^{1/3}}\widehat{\mathbb{I}}_0^{-1}(t_2) = \frac{4\sqrt{1-e^2}\varOmega}{e^4}t_1 - \frac{2(e^2-2)\varOmega}{e^4}t_2.$$

From these expressions, it follows easily that

$$\Lambda(F,\mu)(t_1) = -\frac{e^4+4e^2-8}{2e^4}\varOmega t_3 + \frac{4\sqrt{1-e^2}}{e^4}\varOmega t_4,$$
$$\Lambda(F,\mu)(t_2) = \frac{-4\sqrt{1-e^2}}{e^4}\varOmega t_3 + \frac{e^4+4e^2-8}{2e^4}\varOmega t_4,$$
$$\Lambda(F,\mu)(t_3) = \frac{e^4+4e^2-8}{2e^4}\varOmega t_1 - \frac{4\sqrt{1-e^2}}{e^4}\varOmega t_2,$$
$$\Lambda(F,\mu)(t_4) = \frac{4\sqrt{1-e^2}}{e^4}\varOmega t_1 - \frac{e^4+4e^2-8}{2e^4}\varOmega t_2.$$

Finally, the entries of the Arnold matrix are given by

$$\mathrm{Ar}(t_i,t_j) = \big\langle \mu, \mathrm{ad}_{t_i}\Lambda(F,\mu)(t_j)\big\rangle = \frac{2\sqrt{2}T\varOmega}{(1-e^2)^{1/3}}\big\langle p, \mathrm{ad}_{t_i}\Lambda(F,\mu)(t_j)\big\rangle, \quad i=1,2,3,4.$$

Using (50), the Arnold matrix is then given by

$$\mathrm{Ar} = \begin{bmatrix} A_1 & -A_2 & 0 & 0 \\ -A_2 & A_1 & 0 & 0 \\ 0 & 0 & A_1 & -A_2 \\ 0 & 0 & -A_2 & A_1 \end{bmatrix} \quad \text{with} \quad \begin{cases} A_1 = \dfrac{(8-e^4-4e^2)T\varOmega^2}{e^4(1-e^2)^{\frac{1}{3}}}, \\ A_2 = \dfrac{8(1-e^2)^{\frac{1}{6}}T\varOmega^2}{e^4}. \end{cases}$$

The trace and the determinant of each block of Ar are positive and so Ar is positive definite. □

The next theorem gives the stability of the MacLaurin spheroids.

Theorem 6.2 *A MacLaurin spheroid with eccentricity* $e \in (0,1)$ *and momentum* μ *is nonlinearly* G_μ*-stable if* $S_1 > 0$ *and* $S_2 > 0$, *where*

$$S_1 = \frac{2R}{e^5}\big(9e\big(3-5e^2+2e^4\big) - \sqrt{1-e^2}\big(27-36e^2+8e^4\big)\arcsin e\big),$$
$$S_2 = \frac{R}{e^5}\big(e\big(1-e^2\big)\big(3+4e^2\big) - \sqrt{1-e^2}\big(3+2e^2-4e^4\big)\arcsin e\big).$$

The MacLaurin spheroid is unstable otherwise.

Proof As the Arnold form is nondegenerate, G_μ-stability will follow whenever $(\mathbf{d}_F^2 V_{\xi^\perp} + \mathrm{corr}_{\xi^\perp}(F))|_{\Sigma_{\mathrm{int}}}$ is positive definite. Recall from (12) that

$$\begin{aligned}\Sigma_{\mathrm{int}} = \big\{&\gamma_{\mathrm{SL}(3)}(F) + A : \ \gamma \in \mathfrak{q}^\mu,\ A \in \mathbf{S}^{\mathrm{SL}(3)},\\ &\text{and } \big(\mathbf{D}\mathbb{I}(F)\cdot\big(\gamma_{\mathrm{SL}(3)}(F)+A\big)\big)\big(\xi^\perp\big) \in \mathfrak{p}^*\big\},\end{aligned}$$

where $\mathbf{S}^{\mathrm{SL}(3)}$ is the linear slice for the G-action on SL(3) at the oblate configuration, $F = \mathrm{diag}(a, a, c)$. Matrices $A \in \mathbf{S}^{\mathrm{SL}(3)}$ must verify

$$\begin{aligned}&0 = \mathrm{tr}[A^T(\widehat{\xi}F - F\widehat{\eta})], \quad \forall \xi,\eta \in \mathbb{R}^3, \quad \text{and}\\ &0 = \mathrm{tr}(F^{-1}A),\end{aligned}$$

because A must belong respectively to the orthogonal complement to the tangent space to the group orbit through F and $A \in T_F\mathrm{SL}(3)$. These two conditions give that A must be of the form

$$A = \begin{bmatrix} a_{11} & a_{12} & 0\\ a_{12} & a_{22} & 0\\ 0 & 0 & a_{33}\end{bmatrix} \quad \text{with} \quad \frac{1}{a}(a_{11}+a_{22}) + \frac{1}{c}a_{33} = 0.$$

Therefore, we can describe $\mathbf{S}^{\mathrm{SL}(3)}$ as the set of matrices of the form

$$A = \begin{bmatrix} a_1 + a_2 & a_3 & 0\\ a_3 & a_1 - a_2 & 0\\ 0 & 0 & -2\frac{c}{a}a_1\end{bmatrix}, \tag{51}$$

with $a_1, a_2, a_3 \in \mathbb{R}$. Let the vector $\gamma \in \mathfrak{q}^\mu = \mathfrak{t}$ be $\gamma = (\gamma_1, \gamma_2, \gamma_3, \gamma_4)$ with respect to the basis (t_1, t_2, t_3, t_4). Therefore, using (8), we have

$$\gamma_{\mathrm{SL}(3)}(F) + A = \begin{bmatrix} a_1 + a_2 & a_3 & c\gamma_3 - a\gamma_4\\ a_3 & a_1 - a_2 & -c\gamma_1 + a\gamma_2\\ -a\gamma_3 + c\gamma_4 & a\gamma_1 - c\gamma_2 & -2\frac{c}{a}a_1\end{bmatrix}.$$

The set Σ_{int} is precisely the set of matrices $\lambda_{\mathrm{SL}(3)}(F) + A$ for which

$$\big(\mathbf{D}\mathbb{I}(F)\cdot\big(\gamma_{\mathrm{SL}(3)}(F)+A\big)\big)(\xi_L,\xi_R)^\perp \in \mathfrak{p}^* = (\mathfrak{g}_F + \mathfrak{t})^\circ,$$

where $(\mathfrak{h}+\mathfrak{t})^\circ$ denotes the annihilator of $\mathfrak{h}+\mathfrak{t}$. Using $(\xi_L,\xi_R)^\perp = \frac{\Omega}{\sqrt{2}}p$ and $w \in \{h, t_1, t_2, t_3, t_4\}$ and differentiating (24), the computation of

$$\big\langle w, \big(\mathbf{D}\mathbb{I}(F)\cdot\big(\lambda_{\mathrm{SL}(3)}(F)+A\big)\big)(\xi_L,\xi_R)^\perp\big\rangle$$

gives, in terms of the eccentricity e:

$$\big\langle h, \big(\mathbf{D}\mathbb{I}(F)\cdot\big(\gamma_{\mathrm{SL}(3)}(F)+A\big)\big)(\xi_L,\xi_R)^\perp\big\rangle = 0,$$

$$\begin{aligned}
&\left\langle t_1, \left(\mathbf{D}\mathbb{I}(F)\cdot\left(\gamma_{\mathrm{SL}(3)}(F)+A\right)\right)(\xi_L,\xi_R)^{\perp}\right\rangle \\
&\quad = \frac{T\Omega}{2(1-e^2)^{1/3}}\left[\left(2+e^2\right)\gamma_3 - 2\sqrt{1-e^2}\gamma_4\right], \\
&\left\langle t_2, \left(\mathbf{D}\mathbb{I}(F)\cdot\left(\gamma_{\mathrm{SL}(3)}(F)+A\right)\right)(\xi_L,\xi_R)^{\perp}\right\rangle \\
&\quad = \frac{T\Omega}{2(1-e^2)^{1/3}}\left[2\sqrt{1-e^2}\gamma_3 - \left(2+e^2\right)\gamma_4\right], \\
&\left\langle t_3, \left(\mathbf{D}\mathbb{I}(F)\cdot\left(\gamma_{\mathrm{SL}(3)}(F)+A\right)\right)(\xi_L,\xi_R)^{\perp}\right\rangle \\
&\quad = \frac{T\Omega}{2(1-e^2)^{1/3}}\left[-\left(2+e^2\right)\gamma_1 + 2\sqrt{1-e^2}\gamma_2\right], \\
&\left\langle t_4, \left(\mathbf{D}\mathbb{I}(F)\cdot\left(\gamma_{\mathrm{SL}(3)}(F)+A\right)\right)(\xi_L,\xi_R)^{\perp}\right\rangle \\
&\quad = \frac{T\Omega}{2(1-e^2)^{1/3}}\left[-2\sqrt{1-e^2}\gamma_1 + \left(2+e^2\right)\gamma_2\right].
\end{aligned}$$

It follows from the above expressions that

$$\left(\mathbf{D}\mathbb{I}(F)\cdot\left(\lambda_{\mathrm{SL}(3)}(F)+A\right)\right)(\xi_L,\xi_R)^{\perp} \in (\mathfrak{g}_F+\mathfrak{t})^{\circ}$$

if and only if $\gamma=0$. This is equivalent to $\Sigma_{\mathrm{int}}=\mathbf{S}^{\mathrm{SL}(3)}$. Let us now compute the correction term restricted to $\Sigma_{\mathrm{int}}=\mathbf{S}^{\mathrm{SL}(3)}$. For any $A\in\mathbf{S}^{\mathrm{SL}(3)}$, one has $(\mathbf{D}\mathbb{I}(F)\cdot A)(\xi_L,\xi_R)^{\perp}=\frac{4T\Omega\sqrt{2}}{(1-e^2)^{1/6}}a_1 p$ and so

$$\mathbb{P}_{\mathfrak{t}^*\oplus\mathfrak{p}^*}\left[\left(\mathbf{D}\mathbb{I}(F)\cdot A\right)(\xi_L,\xi_R)^{\perp}\right]=\left(\frac{4T\Omega\sqrt{2}}{(1-e^2)^{1/6}}a_1,0,0,0,0\right).$$

From the expression of $\widehat{\mathbb{I}}_0$, it is straightforward to obtain

$$\widehat{\mathbb{I}}_0^{-1}\left(\mathbb{P}_{\mathfrak{t}^*\oplus\mathfrak{p}^*}\left[\left(\mathbf{D}\mathbb{I}(F)\cdot B\right)(\xi_L,\xi_R)^{\perp}\right]\right)=\sqrt{2}\Omega\left(1-e^2\right)^{1/6}b_1 p,$$

where b_1 is the entry of $B\in\mathbf{S}^{\mathrm{SL}(3)}$ playing the same role of a_1 in A. Then from (13), we have

$$\mathrm{corr}_{(\xi_L,\xi_R)^{\perp}}(F)(A,B)=\left\langle\frac{4T\Omega\sqrt{2}}{(1-e^2)^{1/6}}a_1 p,\sqrt{2}\Omega\left(1-e^2\right)^{1/6}b_1 p\right\rangle=8T\Omega^2 a_1 b_1.$$

The computation of $\mathbf{d}_F^2 V^{\lambda}_{(\xi_L,\xi_R)^{\perp}}(A,B)$ is lengthy but with no difficulties. Using Lemma 6.1, we obtain

$$\begin{aligned}
\left\langle\left(\xi_L,\xi_R\right)^{\perp},\left(\mathbf{D}_F^2\mathbb{I}(A,B)\right)(\xi_L,\xi_R)^{\perp}\right\rangle &= 4a_1 b_1 T\Omega^2, \\
\mathbf{d}_F^2\det(A,B) &= -2c(3a_1b_1+a_2b_2+a_3b_3).
\end{aligned}$$

We fix a basis for the slice $\mathbf{S}^{\mathrm{SL}(3)}$ in which the coordinates of A in (51) are $A=(a_1,a_2,a_3)$. With respect to this basis, the matrix for $\mathbf{d}_F^2 V|_{\Sigma_{\mathrm{int}}}$ is given by $\mathbf{d}_F^2 V|_{\Sigma_{\mathrm{int}}}=$

$\operatorname{diag}(D_1, D_2, D_2)$ with

$$\begin{aligned}\frac{a^{10}}{4}D_1 &= a^4\big(a^6+2\big)V_1+3a^6\big(a^6-1\big)V_2+4\big(a^6-1\big)^2V_{11}+8a^2\big(a^6-1\big)^2V_{12}\\&\quad+4a^4\big(a^6-1\big)^2V_{22},\\\frac{a^{10}}{4}D_2 &= a^{10}V_1+a^6\big(1-a^6\big)V_2.\end{aligned}$$

Therefore, with respect to this basis, we have

$$\begin{aligned}\big(\mathbf{d}^2_F V^\lambda_{\xi^\perp}+\operatorname{corr}_{\xi^\perp}(F)\big)\big|_{\Sigma_{\text{int}}} &= \operatorname{diag}\big(D_1+6c\lambda+6T\Omega^2, D_2+2c\lambda, D_2+2c\lambda\big)\\&= \operatorname{diag}(S_1, S_2, S_2).\end{aligned}\tag{52}$$

For the MacLaurin spheroid we have, from Theorem 5.1, $\lambda = 2(c^2V_1+2cV_2)$ and $\Omega^2 = \frac{2}{T}e^2(V_1+a^2V_2)$. Then in terms of the eccentricity, we have

$$\begin{aligned}S_1 &= \frac{8}{1-e^2}\big(\big(3-4e^2+e^4\big)V_1+3\big(1-e^2\big)^{2/3}V_2+2e^4\big(1-e^2\big)^{2/3}V_{11}\\&\quad+4e^4\big(1-e^2\big)^{1/3}V_{12}+2e^4V_{22}\big),\\S_2 &= \frac{4}{1-e^2}\big(\big(2-3e^2+e^4\big)V_1+\big(2-3e^2\big)\big(1-e^2\big)^{2/3}V_2\big).\end{aligned}$$

Expressing the partial derivatives of V in terms of the integrals $J_O(k,r)$, we obtain

$$\begin{aligned}S_1 &= \frac{2R}{e^5}\big(9e\big(3-5e^2+2e^4\big)-\sqrt{1-e^2}\big(27-36e^2+8e^4\big)\arcsin e\big),\\S_2 &= \frac{R}{e^5}\big(e\big(1-e^2\big)\big(3+4e^2\big)-\sqrt{1-e^2}\big(3+2e^2-4e^4\big)\arcsin e\big).\end{aligned}$$

In order to prove the instability claim, we now study the spectrum of the linearized Hamiltonian vector field $L_h = \omega_N^{-1}(\mathbf{d}^2_{p_x}h_{(\xi_L,\xi_R)^\perp}|_N)$, where in our case N and ω_N are as in Theorem 3.2 and p_x is the point in phase space corresponding to the MacLaurin spheroid. Also, from Proposition 6.2 in Rodríguez-Olmos (2006), we have

$$\mathbf{d}^2_{p_x}h_{(\xi_L,\xi_R)^\perp}|_N = \begin{bmatrix}\mathrm{Ar} & 0 & 0\\ 0 & (\mathbf{d}^2_F V^\lambda_{(\xi_L,\xi_R)^\perp}+\operatorname{corr}_{(\xi_L,\xi_R)^\perp})|_{\mathbf{S}^{\mathrm{SL}(3)}} & 0\\ 0 & 0 & \langle\langle\cdot,\cdot\rangle\rangle_{\mathbf{S}^{\mathrm{SL}(3)*}}\end{bmatrix}.$$

Let us start by computing the block $\langle\langle\cdot,\cdot\rangle\rangle_{\mathbf{S}^{\mathrm{SL}(3)*}}$ of the above matrix, where $\langle\langle\cdot,\cdot\rangle\rangle_{\mathbf{S}^{\mathrm{SL}(3)*}}$ is the inner product on the dual of $\mathbf{S}^{\mathrm{SL}(3)}$ induced by the Riemannian metric on $\mathbf{S}^{\mathrm{SL}(3)}$. For, let as before fix the ordered basis $\{s_1, s_2, s_3\}$ on the slice $\mathbf{S}^{\mathrm{SL}(3)}$

where

$$s_1=\begin{bmatrix}1&0&0\\0&1&0\\0&0&-2\sqrt{1-e^2}\end{bmatrix},\qquad s_2=\begin{bmatrix}1&0&0\\0&-1&0\\0&0&0\end{bmatrix},\qquad s_3=\begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix}.$$

Recalling that $\langle\langle A,B\rangle\rangle_{\mathbf{S}^{\mathrm{SL}(3)}}=T\,\mathrm{tr}(A^TB)$ then the matrix that represents $\langle\langle\cdot,\cdot\rangle\rangle_{\mathbf{S}^{\mathrm{SL}(3)}}$ in the fixed basis is

$$\mathcal{R}_1=2T\begin{bmatrix}3-2e^2&0&0\\0&1&0\\0&0&1\end{bmatrix}.$$

Let $\{s_1^*,s_2^*,s_3^*\}$ be the dual basis of $\{s_1,s_2,s_3\}$ under the identification of $\mathbf{S}^{\mathrm{SL}(3)*}$ with $\mathbf{S}^{\mathrm{SL}(3)}$ using the inner product $\langle\langle\cdot,\cdot\rangle\rangle_{\mathbf{S}^{\mathrm{SL}(3)}}$. In this basis, the induced inner product $\langle\langle\cdot,\cdot\rangle\rangle_{\mathbf{S}^{\mathrm{SL}(3)*}}$ is represented by $\mathcal{R}_1^{-1}$.

Let $\mathcal{R}_2=(\mathbf{d}_F^2V^\lambda_{(\xi_L,\xi_R)^\perp}+\mathrm{corr}_{(\xi_L,\xi_R)^\perp}(F))|_{\Sigma_{\mathrm{int}}}$, then

$$\mathbf{d}^2_{p_x}h_{(\xi_L,\xi_R)^\perp}|_N=\begin{bmatrix}\mathrm{Ar}&0&0\\0&\mathcal{R}_2&0\\0&0&\mathcal{R}_1^{-1}\end{bmatrix}.$$

To compute L_h in the basis $\{t_1,t_2,t_3,t_4,s_1,s_2,s_3,s_1^*,s_2^*,s_3^*\}$ for N we use the formula for ω_N given in Theorem 3.2. Let us now compute each of the blocks of ω_N.

Recall that $\mathfrak{q}^\mu=\mathfrak{t}$ for a MacLaurin spheroid. Then from Theorem 3.2, for $\gamma_1,\gamma_2\in\mathfrak{t}$, we have:

$$\Xi(\gamma_1,\gamma_2)=-\mu\cdot\mathrm{ad}_{\gamma_1}\gamma_2=-\left\langle\frac{2\sqrt{2}T\Omega}{(1-e^2)^{1/3}}p,\mathrm{ad}_{\gamma_1}\gamma_2\right\rangle.$$

Using (50), the matrix Ξ is given by

$$\Xi=\frac{2T\Omega}{(1-e^2)^{1/3}}\begin{bmatrix}0&0&-1&0\\0&0&0&1\\1&0&0&0\\0&-1&0&0\end{bmatrix}.$$

Since for a MacLaurin spheroid $\Sigma_{\mathrm{int}}=\mathbf{S}^{\mathrm{SL}(3)}$, it is immediate from the definition of Ψ in Theorem 3.2 that Ψ is the zero matrix.

We now compute the Coriolis term $-\mathbf{d}\chi^{(\xi_L,\xi_R)^\perp}|_{\Sigma_{\mathrm{int}}}$. For that, we will obtain a concrete expression for the right-hand side of the equality

$$\mathbf{d}\chi^{(\xi_L,\xi_R)^\perp}(X,Y)=X\big(\chi^{(\xi_L,\xi_R)^\perp}(Y)\big)-Y\big(\chi^{(\xi_L,\xi_R)^\perp}(X)\big)-\chi^{(\xi_L,\xi_R)^\perp}\big([X,Y]\big),\quad(53)$$

with $X, Y \in \mathfrak{X}(\mathrm{GL}^+(3))$.

Start by considering for $U, V \in T_I\mathrm{GL}^+(3) = \mathrm{L}(3)$ the corresponding left-invariant extensions $X_U, X_V \in \mathfrak{X}(\mathrm{GL}^+(3))$. We have $X_U(F) = FU$ for every $F \in \mathrm{GL}^+(3)$. Recall that $(\xi_L, \xi_R)^\perp = \frac{\Omega}{2}(\mathbf{e_3}, -\mathbf{e_3})$. Then according to the definition given in Theorem 3.2 we have

$$\begin{aligned}\chi^{(\xi_L,\xi_R)^\perp}(X_U)(F) &= \frac{\Omega T}{2}\operatorname{tr}\big(\big(\widehat{\mathbf{e_3}}F + F\widehat{\mathbf{e_3}}\big)^T FU\big)\\ &= -\frac{\Omega T}{2}\operatorname{tr}\big(\widehat{\mathbf{e_3}}\big(F^TFU + FUF^T\big)\big).\end{aligned} \tag{54}$$

It is straightforward to obtain

$$\begin{aligned}&X_V\big(\chi^{(\xi_L,\xi_R)^\perp}(X_U)\big)(F)\\ &\quad= -\frac{\Omega T}{2}\operatorname{tr}\big(\widehat{\mathbf{e_3}}\big((FV)^TFU + F^TFVU + FVUF^T + FU(FV)^T\big)\big)\\ &\quad= -\frac{\Omega T}{2}\operatorname{tr}\big(\widehat{\mathbf{e_3}}\big(V^TF^TFU + F^TFVU + FVUF^T + FUV^TF^T\big)\big).\end{aligned} \tag{55}$$

Also, since X_U, X_V are left-invariant vector fields, the identity $[X_U, X_V] = X_{UV-VU}$ holds and we have from (54)

$$\chi^{(\xi_L,\xi_R)^\perp}\big([X_U, X_V]\big)(F) = -\frac{\Omega T}{2}\operatorname{tr}\big(\widehat{\mathbf{e_3}}\big(F^TF(UV-VU) + F(UV-VU)F^T\big)\big). \tag{56}$$

In order to compute $-\mathbf{d}\chi^{(\xi_L,\xi_R)^\perp}|_{\Sigma_{\mathrm{int}}}$ let $F = \operatorname{diag}(a, a, c)$ and $A, B \in \Sigma_{\mathrm{int}}$. The unique left-invariant vector fields extending A and B are $X_{F^{-1}A}$ and $X_{F^{-1}B}$, respectively. Then using (53) together with (55) and (56), it is immediate to obtain $-\mathbf{d}\chi^{(\xi_L,\xi_R)^\perp}|_{\Sigma_{\mathrm{int}}} = 0$. Therefore, from Theorem 3.2, the symplectic matrix ω_N and its inverse are

$$\omega_N = \begin{bmatrix}\Xi & 0 & 0\\ 0 & 0 & \mathbf{1}\\ 0 & -\mathbf{1} & 0\end{bmatrix}, \qquad \omega_N^{-1} = \begin{bmatrix}\Xi^{-1} & 0 & 0\\ 0 & 0 & -\mathbf{1}\\ 0 & \mathbf{1} & 0\end{bmatrix}.$$

So, the linearized vector field is

$$L_h = \begin{bmatrix}\Xi^{-1}\mathrm{Ar} & 0 & 0\\ 0 & 0 & -\mathcal{R}_1^{-1}\\ 0 & \mathcal{R}_2 & 0\end{bmatrix},$$

where in our basis $\mathcal{R}_2 = \operatorname{diag}(S_1, S_2, S_2)$ is given in (52).

The block $\Xi^{-1}\mathrm{Ar}$ has imaginary eigenvalues $\epsilon_1^\pm = \pm i\frac{\sqrt{8+e^2}}{2e}\Omega$ with multiplicity 2. For the block,

$$\begin{bmatrix}0 & -\mathcal{R}_1^{-1}\\ \mathcal{R}_2 & 0\end{bmatrix}$$

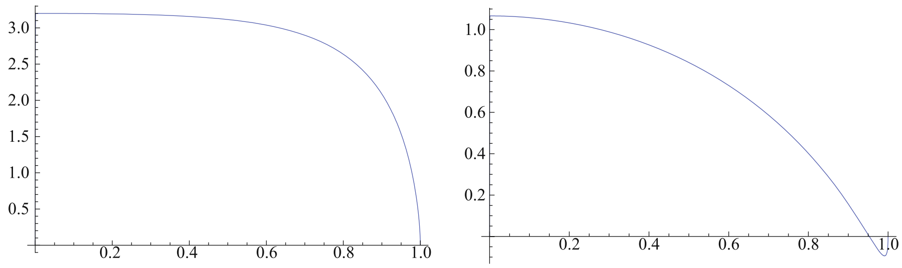

Fig. 1 The plots of S_1 (*left*) and S_2 (*right*) in R units

we obtain the following eigenvalues:

- $\epsilon_2^{\pm} = \pm i\sqrt{\frac{S_1}{(6-4e^2)T}}$ with multiplicity 1, and
- $\epsilon_3^{\pm} = \pm i\sqrt{\frac{S_2}{2T}}$ with multiplicity 2.

As $(6-4e^2) > 0$ for $0 < e < 1$, then $\epsilon_2^{\pm}$ or $\epsilon_3^{\pm}$ become real if and only if S_1 or S_2 become negative, respectively. Hence, if $S_1 < 0$ or $S_2 < 0$, the MacLaurin spheroid becomes linearly unstable, therefore, unstable. This loss of stability corresponds exactly to a collision at 0 of a pair $\epsilon_3^{\pm}$, which passes from being pure imaginary to be real. □

The plots of S_1 and S_2 are shown in Fig. 1. They show that S_1 is always positive for $e \in (0, 1)$ while S_2 has a unique zero, say e_0, being positive for $e < e_0$ and negative for $e > e_0$. We used the Mathematica package system for the numerical computation of e_0 and we obtained $e_0 \simeq 0.952887$. In this case, Theorem 6.2 gives G_μ-stability for the Maclaurin spheroids with eccentricity $e < e_0$.

Remark 6.1

(1) In Riemann's work (Riemann 1861), some conclusions were made concerning the stability of Maclaurin spheroids by studying the existence of a minimum of a certain function G. The existence of this minimum was not done by studying the second variation of G as Riemann says on p. 188: "The direct analysis of the second variation of G when the first variation vanishes would be very complicated; we can however decide if this function has a minimum in the following form:..." He follows with the analysis of the behavior of G. His final conclusion on the stability (ending paragraph 9 of his paper) is the following: "From this study it follows that the case of a rotation of an oblate ellipsoid, around its shortest axis, case already known to MacLaurin, can only be unstable if the relation between the shortest axis with the others is less than 0.303327..." We note that if the relation between the shortest axis of the Maclaurin spheroid and the others is $\frac{c}{a} < 0.303327$; this is equivalent to say that the eccentricity $e > e_0 = 0.952887$.

The value 0.303327 obtained by Riemann follows from his study on the existence of oblate spheroids in pages 184–185 of (Riemann 1861), namely as the root of the last displayed equation in page 184. We remark that this equation is equivalent to the equation $S_2 = 0$ where S_2 is as in Theorem 6.2. Indeed,

$$S_2 = 0 \iff e(1-e^2)(3+4e^2) - \sqrt{1-e^2}(3+2e^2-4e^4)\arcsin e = 0.$$

Taking $e = \cos\psi = \sin(\frac{\pi}{2} - \psi)$ and $\psi \in (0, \frac{\pi}{2})$, one has

$$\begin{aligned} S_2 = 0 \iff & \cos\psi \sin^2\psi\bigl(5 + 2\cos(2\psi)\bigr) \\ & - \sin\psi\left(\frac{5}{2} - \cos(2\psi) - \frac{1}{2}\cos(4\psi)\right)\left(\frac{\pi}{2} - \psi\right) = 0 \\ \iff & 10\sin(2\psi) + 2\sin(4\psi) \\ & + \bigl(-5 + 2\cos(2\psi) + \cos(4\psi)\bigr)(\pi - 2\psi) = 0. \end{aligned} \tag{57}$$

Equation (57) is the same one appearing in Riemann's paper.

Riemann does not present any analytic proof for the existence of a unique root of $S_2 = 0$ in $(0, 1)$, neither any hint on the result that enables its numerical computation. Indeed, after displaying (57), he concludes: "…this equation has, for ψ between 0 and $\pi/2$, the unique root $\sin\psi = 0.303327\ldots$"

(2) For Riemann, the concept of stability was based on the existence of a minimum of a certain Liapunov function. He even considered as being always unstable the relative equilibria that were not a minimum of the Liapunov function. Alternatively, Chandrasekhar considered as "stable" those ellipsoids that were stable in the spectral sense. These different stability notions lead to Chandrasekhar (1987) and Lebovitz (1966) to consider as erroneous some of Riemann's results in particular with respect to some ellipsoids with three distinct axes (see p. 10 and 187 of Chandrasekhar 1987). In fact, Riemann does not prove that the MacLaurin spheroids are unstable for $e > e_0$ as we do in Theorem 6.2.

(3) Liapunov (1904) and independently Poincaré (1885) used Lamé functions for studying the stability of relative equilibria of self-gravitating fluid masses under the hypothesis of preservation of the ellipsoidal shape, but dropping the assumption on the linear dependence of the velocities, (therefore, not fulfilling the conditions of Dirichlet's problem). Liapunov (1904) shows that if the linearity assumption on the velocities is dropped the MacLaurin spheroid is only stable for $e < e_1$ with $e_1 = 0.8126\ldots$ (see p. 11, 61–63 of Liapunov 1904). The point e_1 is exactly the point where the family of MacLaurin spheroids bifurcates into a branch of ellipsoids with 3 distinct axes lengths (Jacobi and Dedekind ellipsoids). He also acknowledges that Riemann's result is correct if one considers Dirichlet's hypotheses. Chandrasekhar (p. 84 of Chandrasekhar 1987) also remarks that at the bifurcation point $e = e_1$ "instability can be induced if some dissipative mechanism is operative." It has been later noted in Chandrasekhar (1987) and Lewis (1993) that MacLaurin spheroids persist to triaxial ellipsoids of type S not only at e_1 but in all the range $0 \le e \le e_0$ of nonlinear stability.

Remark 6.2 The existence of continuous isotropy for the MacLaurin spheroid also plays a role in the spectral analysis of the linearized Hamiltonian vector field and could produce significative differences in the motion of the eigenvalues as the eccentricity varies. Indeed, we have seen that all the eigenvalues remain imaginary for

$e < e_0$, but an imaginary doublet of multiplicity two becomes real for $e > e_0$. This behavior is a consequence of our choice of the orthogonal velocity, which is in turn a consequence of the choice of the splitting of $\mathfrak{g}$ done at the beginning of Sect. 6.2. This motion of eigenvalues trough e_0 coincides with the one obtained by Chandrasekhar (1987) in (108) of Chap. 5. Note that Chandrasekhar's choice for the angular velocity and vorticity in the referred equation are $\frac{1}{2}\Omega$ and $-\frac{1}{2}\Omega$, respectively, exactly as our choice of orthogonal velocity in (47). Earlier in the same chapter of Chandrasekhar (1987), he studies the linearization corresponding to the equivalent MacLaurin spheroid with angular velocity Ω and zero vorticity. In that case, as it follows from the analysis carried out in pp. 81–85 (see, in particular, (50) and Fig. 7a in p. 86) he finds that as e goes through e_0 a double imaginary eigenvalue splits to a complex quadruple with nonzero real and imaginary parts, suggesting a Hamiltonian–Hopf bifurcation. Another difference between our linearization and the one studied by Chandrasekhar is that he linearizes the full Hamiltonian system as viewed from the rotating frame in which the MacLaurin spheroid is an equilibrium. In our approach, the singular reduced energy-momentum method requires only to study the (lower dimensional) linear Hamiltonian system induced in the vector subspace N associated to the augmented Hamiltonian $h_{(\xi_L,\xi_R)^\perp}$. A consequence of this is that in order to conclude instability we only have to study a smaller number of eigenvalues.

6.3 Transversal Spheroids

All the qualitative properties, including stability, of two relative equilibria lying in the same orbit of the symmetry group are the same. Therefore, in view of Remark 5.1, in this subsection, we analyze the nonlinear stability of the + family of transversal spheroids and the main result, Theorem 6.3, will follow for both families. We will set $\omega_+ = \omega$, $f = f_+$ and $(\xi_L, \xi_R)_+ = (\xi_L, \xi_R)$ for notational simplicity. Also, to keep the notation consistent with the proof of Theorem 5.1, we will set $\mathbf{n} = \mathbf{e_2}$.

In this case, the computation of the splitting (10) is simplified due to the fact that $\mathfrak{g}_{p_x} = \{0\}$ and, therefore, $\mathfrak{p} = \mathfrak{g}_\mu$. Introducing the vectors $h = \frac{1}{\sqrt{2}}(\mathbf{e_3}, \mathbf{e_3})$, $p_1 = (\mathbf{e_2}, 0)$, $p_2 = (0, \mathbf{e_2})$, $t_1 = (\mathbf{e_1}, 0)$, $t_2 = (0, \mathbf{e_1})$, $t_3 = \frac{1}{\sqrt{2}}(\mathbf{e_3}, -\mathbf{e_3})$, we choose

$$\begin{aligned}\mathfrak{g}_F &= \operatorname{span}\{h\},\\ \mathfrak{p} &= \operatorname{span}\{p_1, p_2\},\\ \mathfrak{t} &= \operatorname{span}\{t_1, t_2, t_3\}.\end{aligned}$$

These subspaces are obviously invariant under the action of $G_{P_F} = \mathbb{Z}_2(\mathbf{e_2})$. With respect to the basis $(p_1, p_2, t_1, t_2, t_3)$, for $\mathfrak{p} \oplus \mathfrak{t}$, we have

$$\widehat{\mathbb{I}}_0 = T \begin{bmatrix} a^2+c^2 & -2ac & 0 & 0 & 0 \\ -2ac & a^2+c^2 & 0 & 0 & 0 \\ 0 & 0 & a^2+c^2 & -2ac & 0 \\ 0 & 0 & -2ac & a^2+c^2 & \\ 0 & 0 & 0 & 0 & 4a^2 \end{bmatrix}.$$

Or in terms of the eccentricity of a prolate spheroid,

$$\widehat{\mathbb{I}}_0 = T\begin{bmatrix} \frac{2-e^2}{(1-e^2)^{2/3}} & \frac{-2}{(1-e^2)^{1/6}} & 0 & 0 & 0 \\ \frac{-2}{(1-e^2)^{1/6}} & \frac{2-e^2}{(1-e^2)^{2/3}} & 0 & 0 & 0 \\ 0 & 0 & \frac{2-e^2}{(1-e^2)^{2/3}} & \frac{-2}{(1-e^2)^{1/6}} & 0 \\ 0 & 0 & \frac{-2}{(1-e^2)^{1/6}} & \frac{2-e^2}{(1-e^2)^{2/3}} & 0 \\ 0 & 0 & 0 & 0 & 4(1-e^2)^{1/3} \end{bmatrix}.$$

It follows immediately that $\mathfrak{p}$ and $\mathfrak{t}$ are orthogonal with respect to $\widehat{\mathbb{I}}_0$, so our choice of the splitting $\mathfrak{g} = \mathfrak{g}_F \oplus \mathfrak{p} \oplus \mathfrak{t}$ is correct. In this basis, the orthogonal velocity is

$$(\xi_L, \xi_R)^\perp = \omega(p_1 + f p_2),$$

with ω and f as in the $+$ family in Theorem 5.1.

It is straightforward to compute the adjoint representation of $\mathfrak{g}$ in this basis:

Lemma 6.2 *The elements of* $\mathfrak{p} \oplus \mathfrak{t}$ *satisfy the following relations*:

$$\begin{aligned} &\mathrm{ad}_{t_1} t_2 = 0, \qquad \mathrm{ad}_{t_1} t_3 = -\frac{1}{\sqrt{2}} p_1, \qquad \mathrm{ad}_{t_2} t_3 = \frac{1}{\sqrt{2}} p_2, \\ &\mathrm{ad}_{t_1} p_1 = \frac{1}{\sqrt{2}}(h + t_3), \qquad \mathrm{ad}_{t_1} p_2 = 0, \\ &\mathrm{ad}_{t_2} p_1 = 0, \qquad \mathrm{ad}_{t_2} p_2 = \frac{1}{\sqrt{2}}(h - t_3), \\ &\mathrm{ad}_{t_3} p_1 = -\frac{1}{\sqrt{2}} t_1, \qquad \mathrm{ad}_{t_3} p_2 = \frac{1}{\sqrt{2}} t_2, \\ &\mathrm{ad}_{p_1} p_2 = 0. \end{aligned}$$

With this, we can prove the following proposition, which shows that the stability method is applicable.

Proposition 6.2 *The Arnold form for a transversal spheroid is positive definite.*

Proof Recall that the momentum of a transversal spheroid is

$$\mu = \widehat{\mathbb{I}}_0(\xi_L, \xi_R)^\perp = T\omega\big(\kappa^L p_1 + \kappa^R p_2\big),$$

where $\kappa^L := [(a^2+c^2) - (2/a) f]$ and $\kappa^R := [(a^2+c^2) f - (2/a)]$. From Lemma 6.2, and recalling that $\mathrm{ad}^*_\gamma \rho = -\mathrm{ad}_\gamma \rho$, we have, for $\gamma = \gamma^{(1)} t_1 + \gamma^{(2)} t_2 + \gamma^{(3)} t_3 \in \mathfrak{t}$:

$$\mathbb{P}_{\mathfrak{g}_F}\big(\mathrm{ad}^*_\gamma \mu\big) = \frac{-T\omega}{\sqrt{2}}\big(\gamma^{(1)}\kappa^L + \gamma^{(2)}\kappa^R\big) h.$$

This is zero iff $\gamma^{(2)} = -\kappa\gamma^{(1)}$, with

$$\kappa := \frac{\kappa^L}{\kappa^R} = -\frac{(e-1)(e+2)}{(e-2)\sqrt{1-e^2}},$$

and, therefore, $\mathfrak{q}^\mu = \{\gamma^{(1)}(t_1 - \kappa t_2) + \gamma^{(3)} t_3 : (\gamma^{(1)}, \gamma^{(3)}) \in \mathbb{R}^2\}$. A basis for $\mathfrak{q}^\mu$ is given by $\{\gamma_a = t_1 - \kappa t_2, \gamma_b = t_3\}$. Now, proceeding as for the MacLaurin spheroid, we compute

$$\mathrm{ad}_{\gamma_a}\big(\widehat{\mathbb{I}}_0^{-1}\mu\big) = \frac{\omega}{\sqrt{2}}\big((1-\kappa f)h + (1+\kappa f)t_3\big), \qquad \mathrm{ad}_{\gamma_b}\big(\widehat{\mathbb{I}}_0^{-1}\mu\big) = \frac{\omega}{\sqrt{2}}(f t_2 - t_1),$$

$$\mathrm{ad}^*_{\gamma_a}\mu = -\sqrt{2}T\omega\kappa^L t_3, \qquad \mathrm{ad}^*_{\gamma_b}\mu = \frac{T\omega}{\sqrt{2}}\big(\kappa^L t_1 - \kappa^R t_2\big),$$

$$\widehat{\mathbb{I}}_0^{-1}\big(\mathrm{ad}^*_{\gamma_a}\mu\big) = \frac{-\kappa^L\omega}{2\sqrt{2}(1-e^2)^{1/3}} t_3,$$

$$\begin{aligned}\widehat{\mathbb{I}}_0^{-1}\big(\mathrm{ad}^*_{\gamma_b}\mu\big) = {}& \frac{(1-e^2)^{1/6}\omega}{\sqrt{2}e^4}\big(\big(-\sqrt{1-e^2}\big(e^2-2\big)\kappa^L + 2\big(e^2-1\big)\kappa^R\big)t_1 \\ & + \big(\sqrt{1-e^2}\big(e^2-2\big)\kappa^R - 2\big(e^2-1\big)\kappa^L\big)t_2\big).\end{aligned}$$

From where it easily follows that in the basis $\{\gamma_a, \gamma_b\}$ for $\mathfrak{q}^\mu$,

$$\mathrm{Ar} = \mathrm{diag}\left(\frac{3e^4(2+e)T\omega^2}{2(2-e)(1-e^2)^{5/3}}, \frac{4(1+e)(2-e)(2+e)T\omega^2}{e^2(1-e^2)^{2/3}}\right).$$

The entries of Ar are obviously positive. □

We can therefore apply the singular reduced energy-momentum method to study the stability of the transversal spheroid.

Theorem 6.3 *Both families of transversal spheroids are nonlinearly stable for all eccentricities verifying*

$$C_1 = \frac{R}{e^5}\big[-e\big(33-35e^2\big) + \big(33 - 46e^2 + 21e^4\big)\operatorname{arctanh} e\big] > 0$$

and

$$\begin{aligned} C_2 = {}& \frac{R^2}{e^{10}}\big[-27e^2\big(21 - 40e^2 + 19e^4\big) \\ & - 2\big(-567e + 1269e^3 - 831e^5 + 121e^7\big)\operatorname{arctanh} e \\ & - \big(567 - 1458e^2 + 1212e^4 - 334e^6 + 13e^8\big)\operatorname{arctanh}^2 e\big] > 0. \end{aligned}$$

Before proving the theorem, let us remark that the plots of C_1 and C_2, displayed in Fig. 2 show that these quantities are always positive and, therefore, both families of transversal spheroids are stable for all eccentricities.

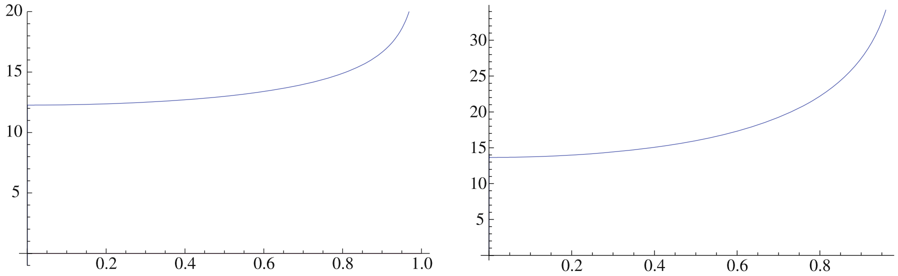

Fig. 2 The plots of C_1 (*left*) and C_2 (*right*), in R and R^2 units, respectively

Proof We start by computing the space of internal variations Σ_{int}. Recall that the slice $\mathbf{S}^{\mathrm{SL}(3)}$ at $F = \operatorname{diag}(a, a, c)$ is given by (51). Hence, for $\gamma = (\gamma^{(1)}, \gamma^{(3)})$ in $\mathfrak{q}^\mu$ and $A = (a_1, a_2, a_3)$ in $\mathbf{S}^{\mathrm{SL}(3)}$ we have

$$\gamma_{\mathrm{SL}(3)}(F) + A = \begin{bmatrix} a_1 + a_2 & a_3 - \sqrt{2}a\gamma^{(3)} & 0 \\ a_3 + \sqrt{2}a\gamma^{(3)} & a_1 - a_2 & -(c + a\kappa)\gamma^{(1)} \\ 0 & (a + c\kappa)\gamma^{(1)} & \frac{-2c}{a}a_1 \end{bmatrix}.$$

Differentiating (24), we find

$$\big\langle h, \big(\mathbf{D}\mathbb{I}(F) \cdot \big(\gamma_{\mathrm{SL}(3)}(F) + A\big)\big)(\xi_L, \xi_R)^\perp\big\rangle = 0,$$

$$\big\langle t_1, \big(\mathbf{D}\mathbb{I}(F) \cdot \big(\gamma_{\mathrm{SL}(3)}(F) + A\big)\big)(\xi_L, \xi_R)^\perp\big\rangle = \frac{2(e+1)T\omega(ea_3 + \sqrt{2}(1-e^2)^{1/6}\gamma^{(3)})}{(1-e^2)^{5/6}},$$

$$\big\langle t_2, \big(\mathbf{D}\mathbb{I}(F) \cdot \big(\gamma_{\mathrm{SL}(3)}(F) + A\big)\big)(\xi_L, \xi_R)^\perp\big\rangle = \frac{-2T\omega(ea_3 + \sqrt{2}(1-e^2)^{1/6}\gamma^{(3)})}{(1-e^2)^{1/3}},$$

$$\big\langle t_3, \big(\mathbf{D}\mathbb{I}(F) \cdot \big(\gamma_{\mathrm{SL}(3)}(F) + A\big)\big)(\xi_L, \xi_R)^\perp\big\rangle = \frac{3\sqrt{2}e^3T\omega\gamma^{(1)}}{(e-2)(1-e^2)^{2/3}}.$$

In order for $(\mathbf{D}\mathbb{I}(F) \cdot (\gamma_{\mathbf{S}^{\mathrm{SL}(3)}} + A))(\xi_L, \xi_R)^\perp \in (\mathfrak{g}_F \oplus \mathfrak{t})^\circ$ all the above expressions must vanish, which happens if and only if $\gamma^{(1)} = 0$ and $\gamma^{(3)} = a_3\epsilon$, with $\epsilon = \frac{-e}{\sqrt{2}(1-e^2)^{1/6}}$. Therefore, we can chose a basis for the space of internal variations Σ_{int} such that any element v belonging to it has components (a_1, a_2, a_3) with the parametrization

$$v = (a_1, a_2, a_3) \mapsto \begin{bmatrix} a_1 + a_2 & a_3(1 - \sqrt{2}a\epsilon) & 0 \\ a_3(1 + \sqrt{2}a\epsilon) & a_1 - a_2 & 0 \\ 0 & 0 & \frac{-2c}{a}a_1 \end{bmatrix}.$$

It is straightforward to obtain

$$\big(\mathbf{D}\mathbb{I}(F)\cdot v\big)(\xi_L,\xi_R)^\perp = 2eT\omega\bigg(\frac{-((e-1)a_1+(e+1)a_2)}{(1-e^2)^{5/6}}p_1 + \frac{(e+1)a_1+a_2(e-1)}{(e-1)(1-e^2)^{1/3}}p_2\bigg).$$

Then the correction term, for $v_1=(a_1,a_2,a_3)$, $v_2=(b_1,b_2,b_3)$, is given by:

$$\mathrm{corr}_{(\xi_L,\xi_R)^\perp}(v_1,v_2) = \frac{8T\omega^2}{e-1}\bigg(\frac{9-5e^2}{e^2-1}a_1b_1 - 3(a_1b_2+a_2b_1) - a_2b_2\bigg).$$

For the restriction of the Hessian of $V^\lambda_{(\xi_L,\xi_R)^\perp}$ at F, we compute first the following second variations

$$\mathbf{d}^2_F V(v_1,v_2) = D_1a_1b_1 + D_2a_2b_2 + D_3a_3b_3,$$

with

$$\begin{aligned}
D_1 &= \frac{4}{(1-e^2)^{5/3}}\big(-\big(1-e^2\big)^{2/3}\big(-3+e^2\big)V_1 + 3e^2\big(e^2-1\big)V_2 + 4e^4V_{11} \\
&\quad + 8e^4\big(1-e^2\big)^{1/3}V_{12} + 4e^4\big(1-e^2\big)^{2/3}V_{22}\big), \\
D_2 &= 4\bigg(V_1 + \frac{e^2}{(1-e^2)^{2/3}}V_2\bigg), \\
D_3 &= \frac{4}{e^2-1}\big(\big(e^4-1\big)V_1 + e^2\big(1-e^2\big)^{1/3}\big(e^2-3\big)V_2\big), \\
&\big\langle(\xi_L,\xi_R)^\perp, \big(\mathbf{D}^2_F\mathbb{I}(v_1,v_2)\big)(\xi_L,\xi_R)^\perp\big\rangle \\
&\quad = 4T\omega^2\bigg(\frac{(9-5e^2)}{(e-1)^2(1+e)}a_1b_1 \\
&\qquad + \frac{3}{1-e}(a_1b_2+a_2b_1) + \frac{1}{(1-e)}a_2b_2 + (1+e)a_3b_3\bigg), \\
\mathbf{d}^2_F\det(v_1,v_2) &= \frac{-2}{(1-e^2)^{1/3}}\big(3a_1b_1 + a_2b_2 + \big(1-e^2\big)a_3b_3\big).
\end{aligned}$$

Putting all the contributions together, and substituting the integral expressions for the derivatives of the potential and the values of λ and ω given in Theorem 5.1, we find that the matrix representing $\mathbf{d}^2_F V^\lambda_{(\xi_L,\xi_R)^\perp} + \mathrm{corr}_{(\xi_L,\xi_R)^\perp}(F)|_{\Sigma_{\mathrm{int}}}$ in the fixed basis of Σ_{int} is block diagonal, with a 2×2 symmetric block in the first two components, and the other block given by the coefficient of a_3b_3, say ϕ, which has the following expression:

$$\phi = \frac{2}{(1-e^2)^{2/3}}\big[\big(1-e^2\big)^{2/3}\big(3+e^2\big)V_1 + \big(3+2e^2-e^4\big)V_2\big].$$

It is clear that ϕ is positive since $(3+2e^2-e^4)$ is positive in (0, 1) and V_1 and V_2 are always positive.

The 2×2 block is $U=\frac{R}{e^5}\left[\begin{smallmatrix} x & y\\ y & z\end{smallmatrix}\right]$ with

$$\begin{aligned}
x &= -27e\left(1-e^2\right)+\left(27-36e^2+17e^4\right)\operatorname{arctanh} e,\\
y &= -9\left(1-e^2\right)\left[3e-(3-e^2)\operatorname{arctanh} e\right],\\
z &= -2e\left(3-4e^2\right)+2\left(3-5e^2+2e^4\right)\operatorname{arctanh} e.
\end{aligned}$$

The matrix U has real eigenvalues. The trace and the determinant of U are:

$$\operatorname{tr}(U)=C_1=\frac{R}{e^5}\left[-e\left(33-35e^2\right)+\left(33-46e^2+21e^4\right)\operatorname{arctanh} e\right]$$

and

$$\begin{aligned}
\det(U)=C_2=\frac{R^2}{e^{10}}\Big[&-27e^2\left(21-40e^2+19e^4\right)\\
&-2\left(-567e+1269e^3-831e^5+121e^7\right)\operatorname{arctanh} e\\
&-\left(567-1458e^2+1212e^4-334e^6+13e^8\right)\operatorname{arctanh}^2 e\Big].
\end{aligned}$$

Therefore, the transversal spheroids are stable whenever C_1 and C_2 are both positive. □

Remark 6.3 As mentioned in Remark 5.3, transversal spheroids have been identified by Chandrasekhar as limiting cases of some families of type S triaxial ellipsoids. In this sense, our stability result gives sharp conditions for the nonlinear stability of the transversal spheroids. To our knowledge, the only stability analysis existing in the literature applicable to the transversal spheroids is the spectral stability conditions found in Chandrasekhar (1987) (see Fig. 15 of that reference, where the transversal spheroids are part of the Riemann's families of S ellipsoids).

7 Conclusions

The main aspects of this work can be summarized as follows:

(1) The singular reduced energy-momentum method of Rodríguez-Olmos (2006) is used as the general framework to study the existence, nonlinear stability, and linear instability of Riemann ellipsoids with symmetric (spheroidal) configurations. The geometric formulation and methods used provide a very concrete type of the nature of the stability and instability results obtained.

(2) Theorem 5.1 provides a complete characterization of all the Riemann ellipsoids with spheroidal configurations existing in Dirichlet's problem. To our knowledge, this is the first time that such a characterization is obtained. In particular, we obtain that there are no spheroidal solutions for which the angular velocity and vorticity are not parallel, as opposed to the case of configurations with three

different axes (see Remark 5.2, case c). This feature of spheroidal configurations does not follow directly from Riemann's theorem.

(3) The stability properties of the MacLaurin spheroids have been studied intensively over the last centuries, often using different stability and instability notions (see Remark 6.1). Although the stability properties of MacLaurin spheroids have been established long time ago, our approach differs from the others in that the singular reduced energy-momentum method enables to obtain simultaneously a block-diagonal normal form for both the Hessian of the augmented Hamiltonian and the linearization of the Hamiltonian vector field. This allows us to conclude (in Theorem 6.2) the precise conditions for both nonlinear stability (in the sense of G_μ-stability as given in Definition 3.1) and instability. Previous approaches to the stability problem of MacLaurin spheroids have obtained conditions either for nonlinear stability but not instability (Riemann 1861), or for spectral stability/instability but not nonlinear stability (Chandrasekhar 1987), or for nonlinear stability but not instability (Lewis 1993).

(4) Theorem 6.3 provides sufficient conditions for the nonlinear stability of transversal spheroid. Figure 2 shows that these spheroids are nonlinearly stable for all values of the eccentricity e. Therefore, they cannot be linearly or spectrally unstable. Hence, we regard Theorem 6.3 as the first time that a complete stability analysis for the family of transversal spheroids is obtained.

(5) In addition, we remark that the derivations of the full stability analysis of the MacLaurin and transversal spheroids shows the power of using geometric methods and the Hamiltonian structure of Dirichlet's problem, and in particular the applicability of the singular reduced energy-momentum method.

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